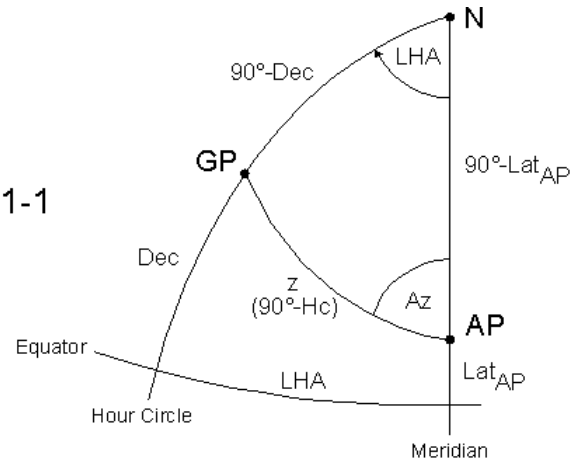
Chapter 11

The Navigational Triangle

The **navigational triangle**, the basis of all common sight reduction procedures, is the (usually oblique) spherical triangle on the earth's surface formed by the north pole, N, the observer's assumed position, AP, and the geographic position of the celestial object, GP, at the time of observation (*Fig. 11-1*).

Fig. 11-1



When using the intercept method, the latitude of the assumed position, Lat_{AP}, the declination of the observed celestial body, Dec, and the local hour angle, LHA, or the meridian angle, t, (calculated from the longitude of AP and the GHA of the object), are initially known to the observer.

The first step is calculating the side z of the navigational triangle by using the law of cosines for sides:

$$\cos z = \cos\left(90° - Lat_{AP}\right) \cdot \cos\left(90° - Dec\right) + \sin\left(90° - Lat_{AP}\right) \cdot \sin\left(90° - Dec\right) \cdot \cos LHA$$

Since cos (90°-x) equals sin x and vice versa, the equation can be written in a simpler form:

$$\cos z = \sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos LHA$$

The side z is not only the **great circle distance** between AP and GP but also the **zenith distance** of the celestial object and the radius of the **circle of equal altitude** (see chapter 1).

Substituting the altitude H for z, we get:

$$\sin H = \sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos LHA$$

Solving the equation for H leads to the altitude formula known from chapter 4:

$$H = \arcsin\left(\sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos LHA\right)$$

The azimuth angle of the observed body is also calculated by means of the law of cosines for sides:

$$\cos\left(90° - Dec\right) = \cos\left(90° - Lat_{AP}\right) \cdot \cos z + \sin\left(90° - Lat_{AP}\right) \cdot \sin z \cdot \cos Az$$

$$\sin Dec = \sin Lat_{AP} \cdot \cos z + \cos Lat_{AP} \cdot \sin z \cdot \cos Az$$

Using the altitude, which has been calculated already, instead of the zenith distance, results in the following equation:

$$\sin Dec = \sin Lat_{AP} \cdot \sin H + \cos Lat_{AP} \cdot \cos H \cdot \cos Az$$

Solving the equation for Az finally yields the azimuth formula from chapter 4:

$$Az = \arccos \frac{\sin Dec - \sin Lat_{AP} \cdot \sin H}{\cos Lat_{AP} \cdot \cos H}$$

The resulting azimuth angle is always in the range of 0°... 180° and therefore not necessarily identical with the true azimuth, Az_N (0°... 360° clockwise from true north) commonly used in navigation. In all cases where sin LHA is negative (GP east of AP) , Az equals Az_N. Otherwise (sin LHA positive, GP westward from AP , as shown in *Fig. 11-1*), Az_N is obtained by subtracting Az from 360°.

When using Sumner's method, z (or H), Dec, and Lat_{AP} (the assumed latitude) are initially known, and LHA (or t) is the quantity to be calculated.

Again, the law of cosines for sides can be applied:

$$\cos z = \cos\left(90° - Lat_{AP}\right) \cdot \cos\left(90° - Dec\right) + \sin\left(90° - Lat_{AP}\right) \cdot \sin\left(90° - Dec\right) \cdot \cos LHA$$

$$\sin H = \sin Lat_{AP} \cdot \sin Dec + \cos Lat_{AP} \cdot \cos Dec \cdot \cos LHA$$

$$\cos LHA = \frac{\sin H - \sin Lat_{AP} \cdot \sin Dec}{\cos Lat_{AP} \cdot \cos Dec}$$

$$LHA = \arccos \frac{\sin H - \sin Lat_{AP} \cdot \sin Dec}{\cos Lat_{AP} \cdot \cos Dec}$$

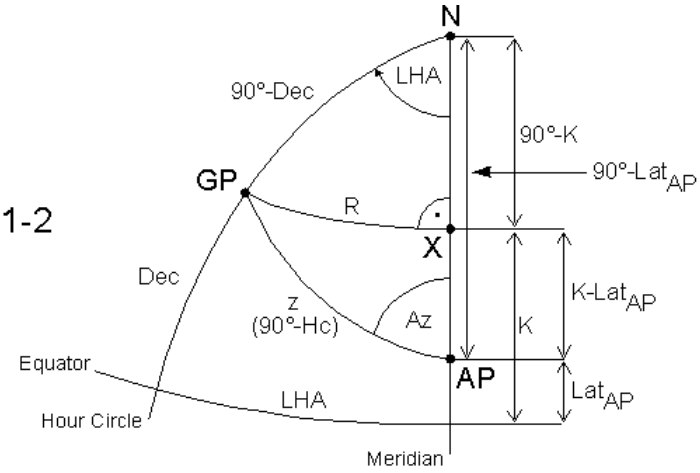
The obtained LHA (or t) is then processed as described in chapter 4.

When observing a celestial body at the time of meridian passage, the local hour angle is zero, and the navigational triangle becomes infinitesimally narrow. In this special case, the formulas of spherical trigonometry are not needed, and the sides of the spherical triangle can be calculated by simple addition or subtraction.

The Divided Navigational Triangle

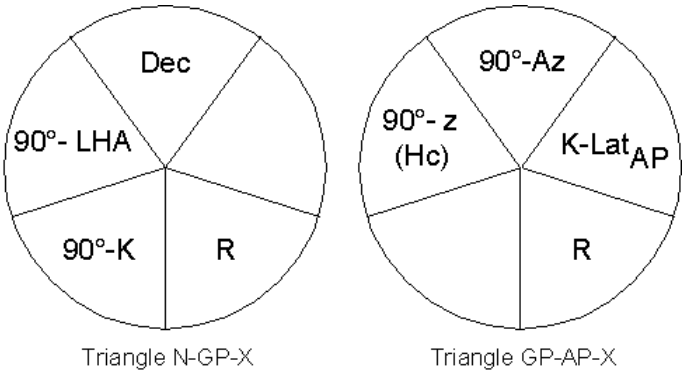
An alternative, very elegant method of solving the navigational triangle begins with dividing it in two right spherical triangles by constructing a great circle passing through GP and intersecting the meridian going through AP at a right angle (*Fig. 11-2*):

Fig. 11-2



The first right triangle is formed by GP, N, and X, the second one by GP, X, and AP. The **auxiliary parts** R and K are intermediate quantities used to calculate z (or Hc) and Az. Both triangles are solved using **Napier's Rules of Circular Parts** (see chapter 9). *Fig. 11-3* illustrates the corresponding circular diagrams:

Fig. 11-3



According to Napier's rules, Hc and Az are calculated by means of the following formulas:

$$\sin R = \sin LHA \cdot \cos Dec \quad \Rightarrow \quad R = \arcsin\left(\sin LHA \cdot \cos Dec\right)$$

$$\sin Dec = \cos R \cdot \sin K \quad \Rightarrow \quad \sin K = \frac{\sin Dec}{\cos R} \quad \Rightarrow \quad K = \arcsin \frac{\sin Dec}{\cos R}$$

$$\sin Hc = \cos R \cdot \cos\left(K - Lat_{AP}\right) \quad \Rightarrow \quad Hc = \arcsin\left[\cos R \cdot \cos\left(K - Lat_{AP}\right)\right]$$

$$\sin R = \cos Hc \cdot \sin Az \quad \Rightarrow \quad \sin Az = \frac{\sin R}{\cos Hc} \quad \Rightarrow \quad Az = \arcsin \frac{\sin R}{\cos Hc}$$

To obtain the true azimuth, Az_N (0°... 360°), the following rules have to be applied:

If the sine of LHA is negative, enter the above formulas with the explementary angle, 360°-LHA, instead of LHA.

$$Az_N = \begin{cases} Az & \text{if} \quad \sin LHA < 0 \text{ AND } Dec > Lat_{AP} \\ 180° - Az & \text{if} \quad \sin LHA < 0 \text{ AND } Dec < Lat_{AP} \\ 180° + Az & \text{if} \quad \sin LHA > 0 \text{ AND } Dec < Lat_{AP} \\ 360° - Az & \text{if} \quad \sin LHA > 0 \text{ AND } Dec > Lat_{AP} \end{cases}$$

The divided navigational triangle is of considerable importance since it forms the theoretical background for a number of **sight reduction tables**, e.g., the Ageton Table.

Using the secant and cosecant function (sec x = 1/cos x, csc x = 1/sin x) and substituting the meridian angle, t, for LHA, the equations for the divided navigational triangle are stated as:

$$\csc R = \csc t \cdot \sec Dec$$

$$\csc K = \frac{\csc Dec}{\sec R}$$

$$\csc Hc = \sec R \cdot \sec\left(K - Lat\right)$$

$$\csc Az = \frac{\csc R}{\sec Hc}$$

In logarithmic form, these equations are stated as:

$$\log\csc R = \log\csc t + \log\sec Dec$$

$$\log\csc K = \log\csc Dec - \log\sec R$$

$$\log\csc Hc = \log\sec R + \log\sec\left(K - Lat\right)$$

$$\log\csc Az = \log\csc R - \log\sec Hc$$

Having the logarithms of the secants and cosecants of angles available in the form of a suitable table, we can solve a sight by a sequence of simple additions and subtractions (beside converting the angles to their corresponding log secants and log cosecants and vice versa). Apart from the table itself, the only tools required are a sheet of paper and a pencil.

The *Ageton* Table (H.O. 211), first published in 1931, is based upon the above formulas and provides a very efficient arrangement of angles and their log secants and log cosecants on 36 pages as well as step by step instructions for use.

Sight reduction tables were developed many years before the invention of electronic calculators in order to simplify calculations necessary to reduce a sight. Still today, sight reduction tables are preferred by people who do not want to deal with the formulas of spherical trigonometry. Moreover, they provide a valuable backup method if electronic devices fail.

A modified version of the *Ageton* Table is available at: http://www.umland.onlinehome.de/page3.htm