Chapter 6

Methods for Latitude and Longitude Measurement

Latitude by Polaris

The observed altitude of a star being vertically above the geographic north pole would be numerically equal to the latitude of the observer (*Fig. 6-1*).

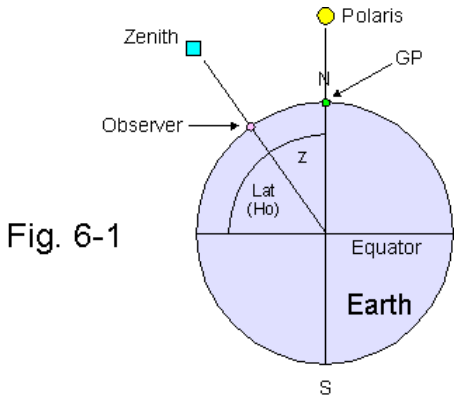


Fig. 6-1

This is nearly the case with the pole star (Polaris). However, since there is a measurable angular distance between Polaris and the polar axis of the earth (presently ca. 1°), the altitude of Polaris is a function of LHA_{Aries}. Nutation, too, influences the altitude of Polaris measurably. To obtain the accurate latitude, several corrections have to be applied:

$$Lat = Ho - 1° + a_0 + a_1 + a_2$$

The corrections a_0, a_1, and a_2 depend on LHA_{Aries}, the observer's estimated latitude, and the number of the month. They are given in the Polaris Tables of the Nautical Almanac [12]. To extract the data, the observer has to know his approximate position and the approximate time.

Noon Latitude (Latitude by Maximum Altitude)

This is a very simple method enabling the observer to determine his latitude by measuring the maximum altitude of an object, particularly the sun. No accurate time measurement is required. The altitude of the sun passes through a flat maximum **approximately** (see noon longitude) at the moment of upper meridian passage (**local apparent noon, LAN**) when the GP of the sun has the same longitude as the observer and is either north or south of him, depending on the observer's geographic latitude. The observer's latitude is easily calculated by forming the algebraic sum or difference of declination and observed zenith distance z (90°-Ho) of the sun. depending on whether the sun is north or south of the observer (*Fig. 6-2*).

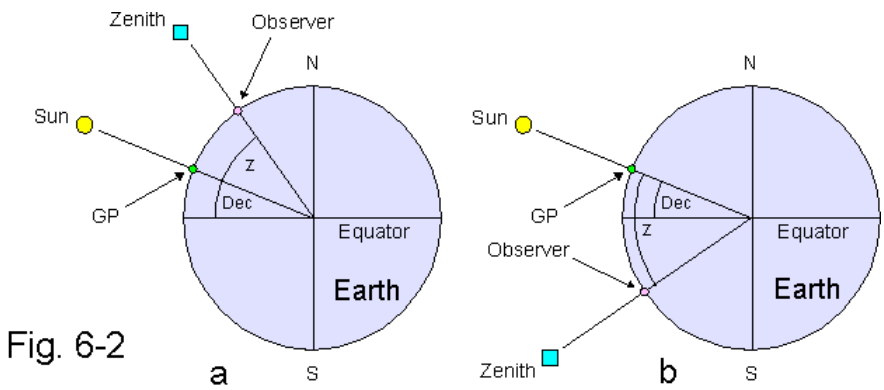


Fig. 6-2

1. Sun south of observer (*Fig. 6-2a*): $$Lat = Dec + (90° - Ho)$$

2. Sun north of observer (*Fig. 6-2b*): $$Lat = Dec - (90° - Ho)$$

Northern declination is positive, southern negative.

Before starting the observations, we need a rough estimate of our current longitude to know the time (GMT) of LAN. We look up the time of Greenwich meridian passage of the sun on the daily page of the Nautical Almanac and add 4 minutes for each degree western longitude or subtract 4 minutes for each degree eastern longitude. To determine the maximum altitude, we start observing the sun approximately 15 minutes before LAN. We follow the increasing altitude of the sun with the sextant, note the maximum altitude when the sun starts descending again, and apply the usual corrections.

We look up the declination of the sun at the approximate time (GMT) of local meridian passage on the daily page of the Nautical Almanac and apply the appropriate formula.

Historically, noon latitude and latitude by Polaris are among the oldest methods of celestial navigation.

Ex-Meridian Sight

Sometimes, it may be impossible to measure the maximum altitude of the sun. For example, the sun may be obscured by a cloud at this moment. If we have a chance to measure the altitude of the sun a few minutes before or after meridian transit, we are still able to find our exact latitude by reducing the observed altitude to the meridian altitude, provided we know our exact longitude (see below) and have an estimate of our latitude.

First, we need the time of local meridian transit (eastern longitude is positive, western longitude negative):

$$T_{Transit}[GMT] = 12 - EoT[h] - \frac{Lon[°]}{15}$$

The meridian angle of the sun, t, is calculated from the time of observation:

$$t[°] = 15 \cdot (T_{Observation}[GMT] - T_{Transit}[GMT])$$

Starting with our estimated Latitude, Lat_E, we calculate the altitude of the sun at the time of observation. We use the altitude formula from chapter 4:

$$Hc = \arcsin(\sin Lat_E \cdot \sin Dec + \cos Lat_E \cdot \cos Dec \cdot \cos t)$$

We further calculate the altitude of the sun at meridian transit, H_{MTC}:

$$H_{MTC} = 90° - |Lat_E - Dec|$$

The difference between H_{MTC} and Hc is called **reduction**, **R**:

$$R = H_{MTC} - Hc$$

Adding R to the **observed altitude**, **Ho**, we get approximately the altitude we would observe at meridian transit, H_{MTO}:

$$H_{MTO} \approx Ho + R$$

From H_{MTO}, we can calculate our improved latitude, $Lat_{improved}$:

$$Lat_{improved} = Dec \pm \left(90° - H_{MTO}\right)$$

(sun south of observer: +, sun north of observer: –)

The exact latitude is obtained by iteration, i. e., we substitute $Lat_{improved}$ for Lat_E and repeat the calculations until the obtained latitude is virtually constant. Usually, no more than one or two iterations are necessary. The method has a few limitations and requires critical judgement. The meridian angle should be smaller than about one quarter of the expected zenith distance at meridian transit ($z_{MT} = |Lat_E – Dec|$), and the meridian zenith distance should be at least four times greater than the estimated error of Lat_E. Otherwise, a greater number of iterations may be necessary. Dec must **not** lie between Lat_E and the true latitude because the method yields erratic results in such cases. If in doubt, we can calculate with different estimated latitudes and compare the results. For safety reasons, the sight should be discarded if the meridian altitude exceeds approx. 85°. If t is not a small angle (t > 1°), we may have to correct the latitude last found for the change in declination between the time of observation and the time of meridian transit, depending on the current rate of change of Dec.

Noon Longitude (Longitude by Equal Altitudes, Longitude by Meridian Transit)

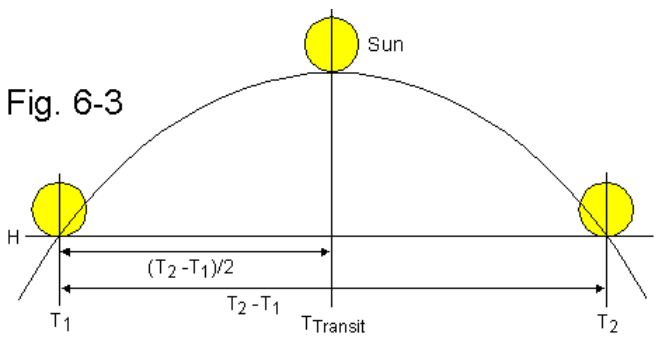
Since the earth rotates with an angular velocity of 15° per hour with respect to the mean sun, the time of local meridian transit (local apparent noon) of the sun, $T_{Transit}$, can be used to calculate the observer's longitude:

$$Lon[°] = 15 \cdot \left(12 - T_{Transit}[h] - EoT_{Transit}[h]\right)$$

$T_{Transit}$ is measured as GMT (decimal format). The correction for EoT at the time of meridian transit, $EoT_{Transit}$, has to be made because the apparent sun, not the mean sun, is observed (see chapter 3). Since the Nautical Almanac contains only values for EoT (see chapter 3) at 0:00 GMT and 12:00 GMT of each day, $EoT_{Transit}$ has to be found by interpolation.

Since the altitude of the sun - like the altitude of any celestial body - passes through a rather flat maximum, the time of peak altitude is difficult to measure. The exact time of meridian transit can be derived, however, from two equal altitudes of the sun.

Assuming that the sun moves along a symmetrical arc in the sky, $T_{Transit}$ is the mean of the times corresponding with a chosen pair of equal altitudes of the sun, one occurring before LAN (T_1), the other past LAN (T_2) (*Fig. 6-3*):



$$T_{Transit} = \frac{T_1 + T_2}{2}$$

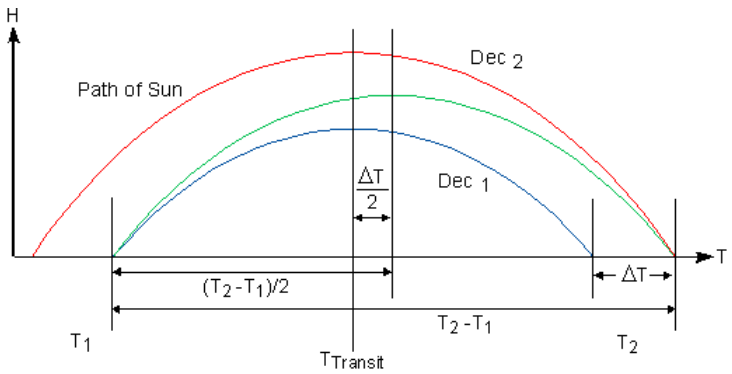
In practice, the times of two equal altitudes of the sun are measured as follows:

In the morning, the observer records the time (T_1) corresponding with a chosen altitude, H. In the afternoon, the time (T_2) is recorded when the descending sun passes through the same altitude again. Since only times of equal altitudes are measured, no altitude correction is required. The interval T_2-T_1 should be greater than 1 hour.

Unfortunately, the arc of the sun is only symmetrical with respect to $T_{Transit}$ if the sun's declination is fairly constant during the observation interval. This is approximately the case around the times of the solstices.

During the rest of the year, particularly at the times of the equinoxes, $T_{Transit}$ differs significantly from the mean of T_1 and T_2 due to the changing declination of the sun. *Fig. 6-4* shows the altitude of the sun as a function of time and illustrates how the changing declination affects the apparent path of the sun in the sky.

Fig. 6-4



The blue line shows the path of the sun for a given, constant declination, Dec_1. The red line shows how the path would look with a different declination, Dec_2. In both cases, the apparent path of the sun is symmetrical with respect to $T_{Transit}$. However, if the sun's declination varies from Dec_1 at T_1 to Dec_2 at T_2, the path shown by the green line will result. Now, the times of equal altitudes are no longer symmetrical to $T_{Transit}$. The sun's meridian transit occurs before $(T_2+T_1)/2$ if the sun's declination changes toward the observer's parallel of latitude, like shown in *Fig. 6-4*. Otherwise, the meridian transit occurs after $(T_2+T_1)/2$. Since time and local hour angle (or meridian angle) are proportional to each other, a systematic error in longitude results.

The error in longitude is negligible around the times of the solstices when Dec is almost constant, and is greatest (up to several arcminutes) at the times of the equinoxes when the rate of change of Dec is greatest (approx. 1 arcminute per hour). Moreover, the error in longitude increases with the observer's latitude and may be quite dramatic in polar regions.

The obtained longitude can be improved, if necessary, by application of the **equation of equal altitudes** [5]:

$$\Delta t \approx \left(\frac{\tan Lat}{\sin t_2} - \frac{\tan Dec_2}{\tan t_2} \right) \cdot \Delta Dec \qquad t_2[°] \approx 15 \cdot \frac{(T_2[h] - T_1[h])}{2}$$

t_2 is the meridian angle of the sun at T_2. Δt is the change in t which cancels the change in altitude resulting from the change in declination between T_1 and T_2, ΔDec.

Lat is the observer's latitude, e. g., a noon latitude. If no accurate latitude is available, an estimated latitude may be used. Dec_2 is the declination of the sun at T_2.

The corrected second time of equal altitude, T_2*, is:

$$T_2^*[h] = T_2[h] - \Delta T[h] = T_2[h] - \frac{\Delta t[°]}{15}$$

At T_2*, the sun would pass through the same altitude as measured at T_1 if Dec did not change during the interval of observation. Accordingly, the time of meridian transit is:

$$T_{Transit} = \frac{T_1 + T_2^*}{2}$$

The correction is very accurate if the exact value for ΔDec is known. Calculating ΔDec with MICA yields a more reliable correction than extracting ΔDec from the Nautical Almanac. If no precise computer almanac is available, ΔDec should be calculated from the daily change of declination to keep the rounding error as small as possible.

Although the equation of equal altitudes is strictly valid only for an infinitesimal change of Dec, dDec, it can be used for a measurable change, ΔDec, (up to several arcminutes) as well without sacrificing much accuracy. Accurate time measurement provided, the residual error in longitude should be smaller than ±0.1' in most cases.

The above formulas are not only suitable to determine one's exact longitude but can also be used to determine the chronometer error if one's exact position is known. This is done by comparing the time of meridian transit calculated from one's longitude with the time of meridian transit derived from the observation of two equal altitudes.

Fig. 6-5 shows that the maximum altitude of the sun is slightly different from the altitude at the moment of meridian passage if the declination changes. Since the sun's hourly change of declination is never greater than approx. 1' and since the maximum of altitude is rather flat, the resulting error of a noon latitude is not significant (see end of chapter).

The equation of equal altitudes is derived from the altitude formula (see chapter 4) using **differential calculus**:

$$\sin H = \sin Lat \cdot \sin Dec + \cos Lat \cdot \cos Dec \cdot \cos t$$

First, we want to know how a small change in declination would affect sin H. We differentiate sin H with respect to Dec:

$$\frac{\partial(\sin H)}{\partial Dec} = \sin Lat \cdot \cos Dec - \cos Lat \cdot \sin Dec \cdot \cos t$$

Thus, the change in sin H caused by an infinitesimal change in declination, d Dec, is:

$$\frac{\partial(\sin H)}{\partial Dec} \cdot d\,Dec = (\sin Lat \cdot \cos Dec - \cos Lat \cdot \sin Dec \cdot \cos t) \cdot d\,Dec$$

Now we differentiate sin H with respect to t in order to find out how a small change in the meridian angle would affect sin H:

$$\frac{\partial(\sin H)}{\partial t} = -\cos Lat \cdot \cos Dec \cdot \sin t$$

The change in sin H caused by an infinitesimal change in the meridian angle, dt, is:

$$\frac{\partial(\sin H)}{\partial t} \cdot d\,t = -\cos Lat \cdot \cos Dec \cdot \sin t \cdot d\,t$$

Since we want both effects to cancel each other, the total differential has to be zero:

$$\frac{\partial(\sin H)}{\partial Dec} \cdot d\,Dec + \frac{\partial(\sin H)}{\partial t} \cdot d\,t = 0$$

$$-\frac{\partial(\sin H)}{\partial t} \cdot d\,t = \frac{\partial(\sin H)}{\partial Dec} \cdot d\,Dec$$

$$\cos Lat \cdot \cos Dec \cdot \sin t \cdot d\,t = \left(\sin Lat \cdot \cos Dec - \cos Lat \cdot \sin Dec \cdot \cos t\right) \cdot d\,Dec$$

$$d\,t = \frac{\sin Lat \cdot \cos Dec - \cos Lat \cdot \sin Dec \cdot \cos t}{\cos Lat \cdot \cos Dec \cdot \sin t} \cdot d\,Dec$$

$$d\,t = \left(\frac{\tan Lat}{\sin t} - \frac{\tan Dec}{\tan t}\right) \cdot d\,Dec$$

Longitude Measurement on a Traveling Vessel

On a traveling vessel, we have to take into account not only the influence of varying declination but also the effects of changing latitude and longitude on sin H during the observation interval. Again, the total differential has to be zero because we want the combined effects to cancel each other with respect to their influence on sin H:

$$\frac{\partial(\sin H)}{\partial t} \cdot (dt + d\,Lon) + \frac{\partial(\sin H)}{\partial\,Lat} \cdot d\,Lat + \frac{\partial(\sin H)}{\partial\,Dec} \cdot d\,Dec = 0$$

$$-\frac{\partial(\sin H)}{\partial t} \cdot (dt + d\,Lon) = \frac{\partial(\sin H)}{\partial\,Lat} \cdot d\,Lat + \frac{\partial(\sin H)}{\partial\,Dec} \cdot d\,Dec$$

Differentiating sin H (altitude formula) with respect to Lat, we get:

$$\frac{\partial(\sin H)}{\partial\,Lat} = \cos Lat \cdot \sin Dec - \sin Lat \cdot \cos Dec \cdot \cos t$$

Thus, the total change in t caused by the combined variations in Dec, Lat, and Lon is:

$$d\,t = \left(\frac{\tan Lat}{\sin t} - \frac{\tan Dec}{\tan t}\right) \cdot d\,Dec + \left(\frac{\tan Dec}{\sin t} - \frac{\tan Lat}{\tan t}\right) \cdot d\,Lat - d\,Lon$$

dLat and dLon are the infinitesimal changes in latitude and longitude caused by the vessel's movement during the observation interval. For practical purposes, we can substitute the measurable changes ΔDec, ΔLat and ΔLon for dDec, dLat and dLon (resulting in the measurable change Δt). ΔLat and ΔLon are calculated from course, C, and velocity, v, over ground and the time elapsed:

$$\Delta Lat['] = v[kn] \cdot \cos C \cdot (T_2[h] - T_1[h])$$

$$\Delta Lon['] = v[kn] \cdot \frac{\sin C}{\cos Lat} \cdot (T_2[h] - T_1[h])$$

$$1\,kn(knot) = 1\,nm/h$$

Again, the corrected second time of equal altitude is:

$$T_2^*[h] = \; = T_2[h] - \frac{\Delta t[°]}{15}$$

The longitude thus calculated refers to T_1. The longitude at T_2 is Lon+ΔLon.

The longitude error caused by changing latitude can be dramatic and requires the navigator's particular attention, even if the vessel moves at a moderate speed.

The above considerations clearly demonstrate that determining one's **exact** longitude by equal altitudes of the sun is not as simple as it seems to be at first glance, particularly on a traveling vessel. It is therefore understandable that with the development of position line navigation (including simple graphic solutions for a traveling vessel) longitude by equal altitudes became less important.

Time Sight

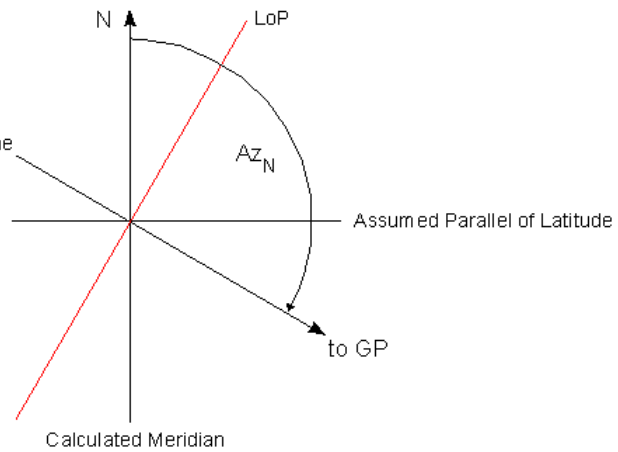
The process of deriving the longitude from a single altitude of a body (as well as the observation made for that purpose) is called **time sight**. However, this method requires knowledge of the exact latitude, e. g., a noon latitude. Solving the altitude formula (chapter 4) for the meridian angle, t, we get:

$$t = \pm \arccos \frac{\sin Ho - \sin Lat \cdot \sin Dec}{\cos Lat \cdot \cos Dec}$$

From t and GHA, we can easily calculate our longitude (see *Sumner's* method, chapter 4). In fact, *Sumner's* method is based upon multiple solutions of a time sight. During a voyage in December 1837, *Sumner* had not been able to determine the exact latitude for several days due to bad weather. One morning, when the weather finally permitted a single observation of the sun, he calculated hypothetical longitudes for three assumed latitudes. Observing that the positions thus obtained lay on a straight line which accidentally coincided with the bearing line of a terrestrial object, he realized that he had found a celestial line of position. This discovery marked the beginning of a new era of celestial navigation.

A time sight can be used to derive a line of position from a single assumed latitude. After solving the time sight, we plot the assumed parallel of latitude and the calculated meridian. Next, we calculate the azimuth of the body with respect to the position thus obtained (azimuth formula, chapter 4) and plot the azimuth line. Our line of position is the perpendicular of the azimuth line going through the calculated position (*Fig. 6-5*).

Fig. 6-5



The latter method is of historical interest only. The modern navigator will certainly prefer the intercept method (chapter 4) which can be used without any restrictions regarding meridian angle (local hour angle), latitude, and declination (see below).

A time sight is not reliable when the body is close to the meridian. Using differential calculus, we can demonstrate that the error in the meridian angle, dt, resulting from an altitude error, dH, varies in proportion with 1/sin t:

$$dt = -\frac{\cos Ho}{\cos Lat \cdot \cos Dec \cdot \sin t} \cdot dH$$

Moreover, dt varies in proportion with 1/cos Lat and 1/cos Dec. Therefore, high latitudes and declinations should be avoided as well. Of course, the same restrictions apply to *Sumner's* method.

The Meridian Angle of the Sun at Maximum Altitude

As mentioned above, the moment of maximum altitude does not exactly coincide with the upper meridian transit of the sun (or any other body) if the declination is changing. At maximum altitude, the rate of change of altitude caused by the changing declination cancels the rate of change of altitude caused by the changing meridian angle. The equation of equal altitude can be used to calculate the meridian angle of the sun at this moment. We divide each side of the equation by the infinitesimal time interval dT:

$$\frac{d\,t}{d\,T} = \left(\frac{\tan Lat}{\sin t} - \frac{\tan Dec}{\tan t} \right) \cdot \frac{d\,Dec}{d\,T}$$

Measuring the rate of change of t and Dec in arcminutes per hour we get:

$$900\,'/h = \left(\frac{\tan Lat}{\sin t} - \frac{\tan Dec}{\tan t} \right) \cdot \frac{d\,Dec\,[']}{d\,T\,[h]}$$

Sine t is very small, we can substitute tan t for sin t:

$$900 \approx \frac{\tan Lat - \tan Dec}{\tan t} \cdot \frac{d\,Dec\,[']}{d\,T\,[h]}$$

Now, we can solve the equation for tan t:

$$\tan t \approx \frac{\tan Lat - \tan Dec}{900} \cdot \frac{d\,Dec\,[']}{d\,T\,[h]}$$

Since a small angle (in radians) is nearly equal to its tangent, we get:

$$t\,[°] \cdot \frac{\pi}{180} \approx \frac{\tan Lat - \tan Dec}{900} \cdot \frac{d\,Dec\,[']}{d\,T\,[h]}$$

Measuring t in arcminutes, the equation is stated as:

$$t\,['] \approx 3.82 \cdot (\tan Lat - \tan Dec) \cdot \frac{d\,Dec\,[']}{d\,T\,[h]}$$

dDec/dT is the rate of change of declination measured in arcminutes per hour.

The maximum altitude occurs after LAN if t is positive, and before LAN if t is negative.

For example, at the time of the spring equinox (Dec = 0, dDec/dT ≈ +1'/h) an observer being at +80° (N) latitude would observe the maximum altitude of the sun at t ≈ +21.7', i. e., 86.7 seconds after meridian transit (LAN). An observer at +45° latitude, however, would observe the maximum altitude at t ≈ +3.82', i. e., only 15.3 seconds after meridian transit.

We can use the last equation to evaluate the systematic error of a noon latitude. The latter is known to be based upon the maximum altitude, not on the meridian altitude of the sun. Following the above example, the observer at 80° latitude would observe the maximum altitude 86.7 seconds after meridian transit. During this interval, the declination of the sun would have changed from 0 to +1.445" (assuming that Dec is 0 at the time of meridian transit). Using the altitude formula (chapter 4), we get:

$$Hc = \arcsin\left(\sin 80° \cdot \sin 1.445'' + \cos 80° \cdot \cos 1.445'' \cdot \cos 21.7'\right) = 10°\ 0'\ 0.72''$$

In contrast, the calculated altitude at meridian transit would be exactly 10°. Thus, the error of the noon latitude would be -0.72".

In the same way, we can calculate the maximum altitude of the sun observed at 45° latitude:

$$Hc = \arcsin\left(\sin 45° \cdot \sin 0.255'' + \cos 45° \cdot \cos 0.255'' \cdot \cos 3.82'\right) = 45°\ 0'\ 0.13''$$

In this case, the error of the noon latitude would be only -0.13".

The above examples show that even at the times of the equinoxes, the systematic error of a noon latitude caused by the changing declination of the sun is much smaller than other observational errors, e. g., the errors in dip or refraction. A significant error in latitude can only occur if the observer is very close to one of the poles (tan Lat!). Around the times of the solstices, the error in latitude is practically non-existent.