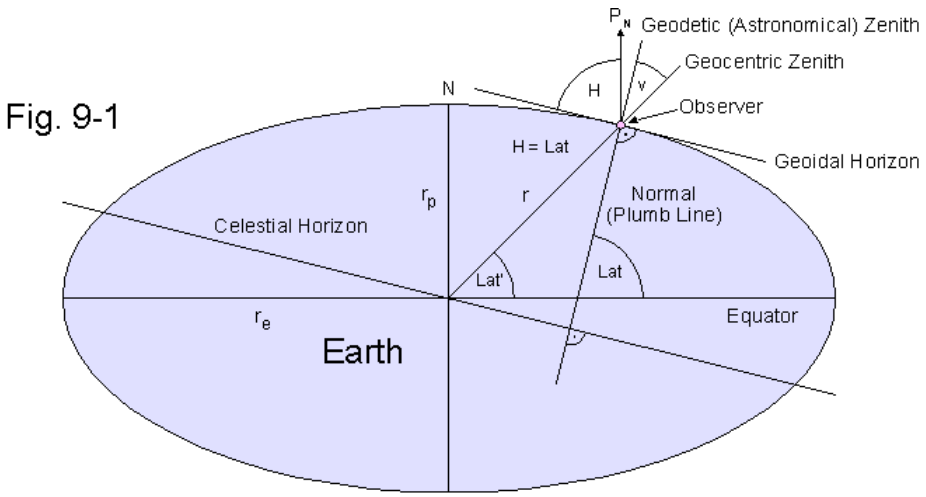
Chapter 9

Geodetic Aspects of Celestial Navigation

The Ellipsoid

Celestial navigation is based upon the assumption that the earth is a sphere and, consequently, on the laws of spherical trigonometry. In reality, the shape of the earth is rather irregular and approximates an **oblate spheroid (ellipsoid)** resulting from two forces, **gravitation** and **centrifugal force**, acting on the viscous body of the earth. While gravitation alone would force the earth to assume the shape of a sphere, the state of lowest potential energy, the centrifugal force caused by the earth's rotation contracts the earth along the polar axis (axis of rotation) and stretches it along the plane of the equator. The local vector sum of both forces is called **gravity**.

A number of reference ellipsoids are in use to describe the shape of the earth, for example the **World Geodetic System (WGS)** ellipsoid of 1984. **The following considerations refer to the ellipsoid model of the earth which is sufficient for most navigational purposes**. *Fig.9-1* shows a meridional section of the ellipsoid.



Earth data (WGS 84 ellipsoid) :

Equatorial radius	r_e	6378137.0 m
Polar radius	r_p	6356752.3142 m
Flattening	$(r_e - r_p) / r_e$	1/298.25722

Due to the flattening of the earth, we have to distinguish between **geographic** and **geocentric** latitude which would be the same if the earth were a sphere. The geographic (geodetic) latitude of a given position, Lat, is the angle formed by the local normal (perpendicular) to the adopted ellipsoid and the plane of the equator. The geocentric latitude, Lat', is the angle formed by the local radius vector and the plane of the equator. Geographic and geocentric latitude are interrelated as follows:

$$\tan Lat' = \frac{r_p^2}{r_e^2} \cdot \tan Lat$$

Geographic and geocentric latitude are equal at the poles and on the equator. At all other places, the geocentric latitude, Lat', is smaller than the geographic latitude, Lat. As with the spherical earth model, geographic and geocentric longitude are equal. **Maps are always based upon geographic (geodetic) coordinates.**

In the following, we will discuss the effects of the oblateness (flattening) of the earth on celestial navigation. Altitudes (or zenith distances) measured by the navigator always refer to the local plumb line which aligns itself with gravity and points to the **astronomical zenith**.

Even the visible sea horizon correlates with the astronomical zenith since the water surface is perpendicular to the local plumb line. Under the assumption of a homogeneous mass distribution throughout the ellipsoid, the plumb line coincides with the local normal to the ellipsoid which points to the **geodetic zenith**. Thus, astronomical and geodetic zenith are identical.

The **geocentric zenith** is defined as the point where the extended local radius vector of the earth intersects the celestial sphere. The angular distance of the astronomical (geodetic) zenith from the **geocentric zenith** is called **angle of the vertical**, **v**. The angle of the vertical is a function of the geographic latitude. The following formula was proposed by *Smart* [9]:

$$v['] \approx 692.666 \cdot \sin(2 \cdot Lat) - 1.163 \cdot \sin(4 \cdot Lat) + 0.026 \cdot \sin(6 \cdot Lat)$$

The coefficients of the formula have been adapted to the proportions of the WGS 84 ellipsoid.

The angle of the vertical at a given position equals the difference between geographic and geocentric latitude (*Fig. 9-1*):

$$v = Lat - Lat'$$

The maximum value of v, occurring at 45° geographic latitude, is approx. 11.5'. Thus, the geocentric latitude of an observer being at 45°geographic latitude is only 44° 48.5'. This difference is not negligible. Therefore, the navigator has to know if the coordinates of a fix obtained by astronomical observations are geographic or geocentric. Altitudes are measured with respect to the sea horizon or an artificial horizon. Both correlate with the local plumb line which points to the astronomical zenith. Thus, the latter is the only reference available to the navigator. As demonstrated in *Fig. 9-1*, the altitude of the celestial north pole, P_N, (corrected altitude of Polaris) with respect to the geoidal horizon equals the geographic, not the geocentric latitude. A noon latitude, being the sum or difference of the (geocentric) declination and the zenith distance with respect to the astronomical zenith would give the same result.

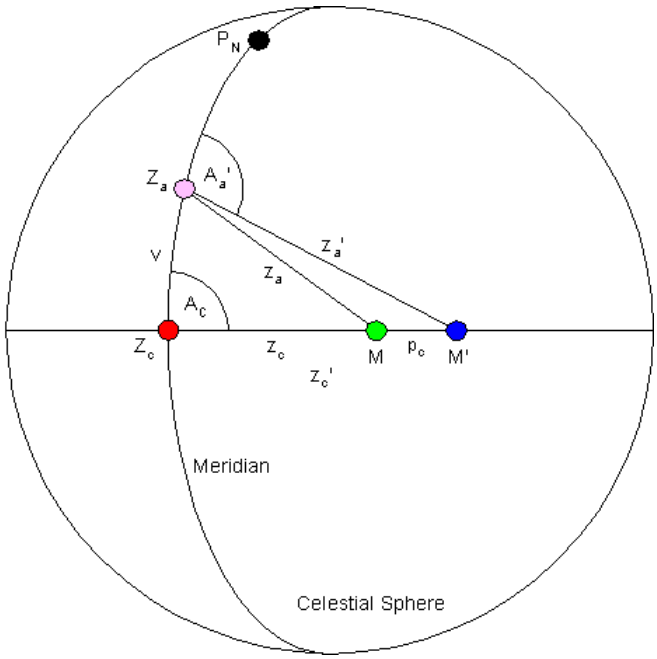
Latitudes obtained by celestial observations are geographic latitudes since the navigator measures altitudes with respect to the local astronomical zenith (directly or indirectly).

It is further important to know if the oblateness of the earth causes significant errors due to the fact that calculations of celestial navigation are based on a spherical earth model. According to the above values for polar radius and equatorial radius of the earth, the great circle distance of 1' is 1.849 km at the poles and 1.855 km at the equator. This small difference does not produce a significant error when plotting lines of position. It is therefore sufficient to use the adopted mean value (1 nautical mile = 1.852 km). However, when calculating the great circle distance (see chapter 11) of two locations thousands of nautical miles apart, the error caused by the oblateness of the earth can increase to several nautical miles. If extraordinary precision is required, the formulas for geodetic distance given in [2] should be used.

The Parallax of the Moon

Due to the oblateness of the earth, the distance between geoidal and celestial horizon is not constant but can assume any value between r_p and r_e, depending on the observer's latitude. This has a measurable effect on the parallax of the moon since tabulated values for HP refer to the equatorial radius, r_e. The parallax of the moon is further affected by the displacement of the plumb line from the earth's center. A correction formula compensating both effects is given in chapter 2. The asymmetry of the plumb line with respect to the earth's center even causes a small (negligible) parallax in azimuth unless the moon is on the local meridian. In the following, we will calculate the effects of the oblateness of the earth on the parallax of the moon with the exact formulas of spherical astronomy [9]. For practical navigation, the simplified correction formulas given in chapter 2 are accurate enough. *Fig.* 9-2 shows a projection of the astronomical zenith, Z_a, the geocentric zenith, Z_c, and the geographic position of the moon, M, on the **celestial sphere**, an imaginary hollow sphere of infinite diameter with the earth at its center.

Fig. 9-2



The geocentric zenith, Z_c, is the point where a straight line from the earth's center through the observer's position intersects the celestial sphere. The astronomical zenith, Z_a, is the point at which the plumb line going through the observer's position intersects the celestial sphere. Z_a and Z_c are on the same meridian. M is the projected geocentric position of the moon defined by Greenwich hour angle and declination. Unfortunately, the position of a body defined by GHA and Dec is commonly called geographic position (see chapter 3) although GHA and Dec are geocentric coordinates. M' is the point where a straight line from the observer through the moon's center intersects the celestial sphere. Z_c, M, and M' are on a great circle. The zenith distance measured by the observer is z_a' because the astronomical zenith is the available reference. The quantity we want to know is z_a, the astronomical zenith distance corrected for parallax in altitude. This is the angular distance of the moon from the astronomical zenith, measured by a fictitious observer at the earth's center.

The known quantities are v, A_a', and z_a'. In contrast to the astronomer, the navigator is usually not able to measure A_a' precisely. For navigational purposes, the calculated azimuth (see chapter 4) may be substituted for A_a'.

We have three spherical triangles, Z_aZ_cM', Z_aZ_cM, and Z_aMM'. First, we calculate z_c' from z_a', v, and A_a' using the **law of cosines for sides** (see chapter 10):

$$\cos z_c^{/} = \cos z_a^{/} \cdot \cos v + \sin z_a^{/} \cdot \sin v \cdot \cos\left(180° - A_a^{/}\right)$$

$$z_c^{/} = \arccos\left(\cos z_a^{/} \cdot \cos v - \sin z_a^{/} \cdot \sin v \cdot \cos A_a^{/}\right)$$

To obtain z_c, we first have to calculate the relative length ($r_e = 1$) of the radius vector, r, and the geocentric parallax, p_c:

$$\frac{r}{r_e} = \sqrt{\frac{1-\left(2e^2 - e^4\right)\cdot \sin^2 Lat}{1 - e^2 \cdot \sin^2 Lat}} \qquad e^2 = 1 - \frac{r_p^2}{r_e^2}$$

$$p_c = \arcsin\left(\frac{r}{r_e} \cdot \sin HP \cdot \sin z_c^{/}\right)$$

HP is the equatorial horizontal parallax.

The geocentric zenith distance corrected for parallax is:

$$z_c = z_c^{/} - p_c$$

Using the cosine formula again, we calculate A_c, the azimuth angle of the moon with respect to the geocentric zenith:

$$A_c = \arccos \frac{\cos z_a^{/} - \cos z_c^{/} \cdot \cos v}{\sin z_c^{/} \cdot \sin v}$$

The astronomical zenith distance corrected for parallax is:

$$z_a = \arccos \left(\cos z_c \cdot \cos v + \sin z_c \cdot \sin v \cdot \cos A_c \right)$$

Thus, the **parallax in altitude** (astronomical) is:

$$PA = z_a^{/} - z_a$$

The small angle between M and M', measured at Z_a, is the **parallax in azimuth**, p_{az}:

$$p_{az} = \arccos \frac{\cos p_c - \cos z_a \cdot \cos z_a^{/}}{\sin z_a \cdot \sin z_a^{/}}$$

Comparing the exact formulas with the simplified parallax formulas given in chapter 2 (including the correction for the oblateness of the earth), we will find very small differences in the observed altitude (small fractions of an arcsecond). These small differences are not measurable with the instruments of celestial navigation and can be neglected.

The parallax in azimuth does not exist when the moon is on the local meridian and is greatest when the moon ist east or west of the observer. It is further greatest at medium latitudes (45°) and non-existant when the observer is at one of the poles or on the equator (v = 0). Even under the most unfavourable conditions, the parallax in azimuth is only a fraction of an arcminute and therefore insignificant to celestial navigation.

Other celestial bodies do not require a correction for the oblateness of the earth since their parallaxes are very small compared with the parallax of the moon.

The Geoid

The earth is not **exactly** an oblate ellipsoid. The shape of the earth is more accurately described by the **geoid**, an equipotential surface of the earth's field of gravity. This surface has elevations and depressions caused by geographic features and a non-uniform mass distribution (materials of different density). Elevations occur at local accumulations of matter (mountains, ore deposits), depressions at local deficiencies of matter (valleys, lakes, caverns). The elevation or depression of each point of the geoid with respect to the ellipsoid is found by gravity measurement. At the flank of an elevation or depression of the geoid, the plumb line (the normal to the geoid) does not coincide with the normal to the ellipsoid, and the astronomical zenith differs from the geodetic zenith. Thus, the astronomical latitude or longitude of a position (obtained through astronomical observations) may slightly differ from the geographic (geodetic) latitude or longitude. These differences are usually smaller than one arcminute, but greater local differences have been reported, for example, in coastal waters with adjacent high mountains.

The errors caused by the irregular shape of the geoid are usually not relevant to celestial navigation at sea but are important to surveying and map-making where a higher degree of accuracy is required.