MATEMATIČNE METODE v fiziki

LITERATURA
- Matematika v fiziki in tehniki (FMF)

Kaj bomo delali?
- Vektorji
- Matrike
- Tenzorji
- Ekstremalni račun
- Vektorska polja
- Taylorjeva vrsta

KOLOKVIJI

Imeli bomo $\boxed{\text{3 pisne vaje}}$ — vsaki 2 nalogi. Vsaka je vredna 1 točko, 3 točke so pozitivno.

ocene:	6	7	8	9	10
točke:	3	3,5	4	4,5	5

POPRAVNI KOLOKVIJ
- 3 naloge iz teh področij → vsaka naloga po 2 točki → reševali bomo lahko le tiste, ki smo imeli v sklopu 50 ali manj %

IZPITI:
- 14. junij 10:00 P006
- 28. junij 10:00 P006
- 13. september 10:00 P006

- Na testu imamo lahko s seboj . Matematični priročnik.
- Ustnega izpita ni.

LITERATURA

- Ivan Vidav (več knjig, po poglavjih)

RAČUNANJE Z FIZIKALNIMI KOLIČINAMI

23. 2. 10

① DIMENZIJE

OPERACIJE: seštevanje (enake enote

A [Nm]
M [Nm] } čeprav imata enako enoto se ne moreta seštevati
- navor je vektor

tenzorji

ZRCALJENJE $\vec{r} \to -\vec{r}$
$\vec{v} \to -\vec{v}$ (tudi hitrost se prezrcali)
$\vec{a} \to -\vec{a}$
$\vec{F} \to -\vec{F}$
} POLARNI VEKTORJI

$\vec{M} = \vec{r} \times \vec{F} \longrightarrow (-\vec{r}) \times (-\vec{F}) = \vec{r} \times \vec{F} \longrightarrow \vec{M}$ → navor ne spremeni predznaka

↘ AKSIALNI VEKTORJI

$s = s_0 + vt + \frac{1}{2}at^2$ → lahko vzamemo več različnih enot

$t = \frac{130}{v_A - v_B}$ → važno kaj vstaviš

$\lambda = 12{,}3\sqrt{u}$ → enote ne štimajo (volt, meter) → nova enota
$\lambda = k\sqrt{u}$ → nujno napišemo s simbol → vemo pa, da je da je $k = 12{,}3\,\text{Å}\,(\text{kV})^{-\frac{1}{2}}$

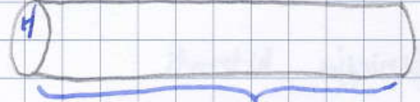
DIMENZIJSKA ANALIZA

- po enotah lahko vidimo kakšna je enačba

$\underset{s^{-1}}{\omega} = \underset{m}{\cancel{rv}}, \underset{m/s}{\cancel{\frac{r}{v}}}, \frac{v}{r}$ ✓

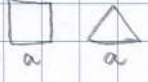
Enote morajo nemo vedno štimat.

PRETAKANJE TEKOČIN PO CEVEH



$$\Phi = a^{\alpha} \cdot p'^{\beta} \cdot \eta^{\gamma}$$

Koeficiente dobojo tako, da enote štimajo



$p' = \frac{\Delta p}{l}$ ↘ sprememba tlaka po dolžini

$$\Phi_v = a^{\alpha} \quad p'^{\beta} \quad \eta^{\gamma}$$

$$m^3 s^{-1} = m^{\alpha} \quad (kg\, s^{-2}\, m^{-2})^{\beta} \quad (kg\, m^{-1} s^{-1})^{\gamma}$$

$$\frac{\Delta p}{l} = \frac{N}{m^3} = \frac{kg\, m}{s^2\, m^3}$$

$$\eta = \frac{kg}{m s}$$

$$m^3 s^{-1} = m^{\alpha}\, kg^{\beta}\, s^{-2\beta}\, m^{-2\beta}\, kg^{\gamma}\, m^{-1\gamma}\, s^{-\gamma}$$

$m: 3 = \alpha - 2\beta - \gamma$

$s: -1 = -2\beta - \gamma$

$kg: 0 = \beta + \gamma \qquad \beta = -\gamma$

$-1 = -2\beta + \beta$
$\beta = 1 \qquad \gamma = -1$

$3 = \alpha - 2 + 1$
$\alpha = 4$

$$\Phi_v = a^4\, p'\, \eta^{-1} \cdot K$$

- odvisna od profila
- neodvisna od prečne velikosti
- določimo z merjenjem

ENAČBE

- definicijska enačba $\vec{v} = \frac{d\vec{r}}{dt}$

$\vec{a} = \frac{d\vec{v}}{dt}$

$\vec{M} = \vec{r} \times \vec{F}$

- zakoni $\vec{a} = \frac{\vec{F}}{m}$
  - vse količine, ki nastopajo so nam bile že znane, definirane
  - to ugotovitev dobimo znanstveno

zakon = aksiom

- izreki $\rightarrow m\vec{a} = \vec{F} \quad | \, dt$

$m\vec{v} - m\vec{v}_0 = \int F dt$ IZREK O GIBALNI KOLIČINI

$\rightarrow s = v \cdot t \qquad v = \frac{s}{t} \Rightarrow$ to (NI) definicija hitrosti

$v = \frac{s}{t} \,|\, dt \qquad \int v\, dt = \int ds \rightarrow v =$ konstanta

- fenomenološke zveze (zveze, ki jih dobimo s poskusi ~~ven~~)

$F = kx$ Hookov zakon

$F = kx + cx^2$

$\Delta l = l\alpha_T \Delta T \rightarrow$ fenomenološka zveza
$\rightarrow$ definicijska enačba (α_T)

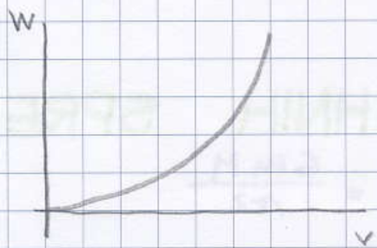
$\alpha_T = \frac{dl}{l\,dT} \rightarrow$ definicija (zapišemo z odvodi)

$\rightarrow$ lahko pa je tudi izrek

Grafi

- za ponazoritev zveze

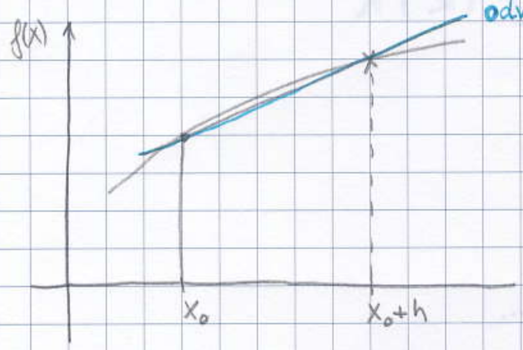
$W_{kin} = \frac{1}{2} m v^2$



Rabimo, ker so pomembna ilustracija naših enačb.

$\left(\frac{\sin ax}{x}\right) \rightarrow$ lim je a

UPORABA ODVODOV



$f(x_0 + h) = ?$
$f(x_0)$
$f'(x_0)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \approx \frac{f(x+h) - f(x)}{h}$$

$$f(x_0+h) = f(x_0) + f'(x_0)h$$

KOTI SO V RADIANIH!

TAYLORJEVA FORMULA

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \ldots + \frac{1}{n!}f^{(n)}(x_0)h^n$$

$x_0 = 0$
$h = x$

$\sin x = ?$

$\sin(0) = 0$
$\sin'(0) = \cos(0) = 1$
$\sin''(0) = \cos'(0) = -\sin(0) = 0$
$\sin'''(0) = \cos''(0) = -\sin'(0) = -\cos(0) = -1$

$$\sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!}$$

Če je kot zelo majhen, lahko sin kot aproksimiramo kar z kotom, ki je nujno v radianih.

2.3.10

a) RAČUNANJE MAJHNIH SPREMEMB

1. PRIMER

GRAVITACIJSKA SILA: $mg = \frac{GmM}{r^2}$

$$g = \frac{GM}{r^2}$$

$r = R$ (radij zemlje) $= 6{,}4 \cdot 10^6 m$

$g(R) = g_0$

Uporabili bomo Taylorjevo vrsto, da bomo lahko izračunal bolj natančno.

$$g(R+h) = g(R) + g'(r)\big|_{r=R} h + \frac{1}{2}g''(R')h^2$$

$$g' = -\frac{2GM}{r^3}$$

$$g'' = +\frac{6GM}{r^4}$$

$$g(R+h) = g_0 - 2\frac{GM}{R^3}h + 3\frac{GM}{R^4} \cdot h^2 + \ldots$$

$$\frac{g(3)}{g(2)} = \frac{3\frac{GM}{R^3}h^2}{2\frac{GM}{R^2}h} = \frac{3h}{2R}$$

$$\Delta g = g(R+h) - g_0 = -\frac{2GM}{R^2}\frac{h}{R} = -2g_0\frac{h}{R}$$

RELATIVNA SPREMEMBA POSPEŠKA: $\frac{\Delta g}{g_0} = -\frac{2h}{R}$

Trik

$$g = \frac{GM}{r^2} \Big| \ln$$

$$\ln g = \ln(G) + \ln(M) - \ln(r^2) = \ln(G) + \ln(M) - 2\ln(r)$$

POIŠČI DIFERENCIAL

$$\frac{1}{g}dg = 0 + 0 - 2\frac{1}{r}dr$$

$$\frac{dg}{g} = -2\frac{dr}{r} \Rightarrow \boxed{\frac{\Delta g}{g} = -2\frac{\Delta r}{r}}$$

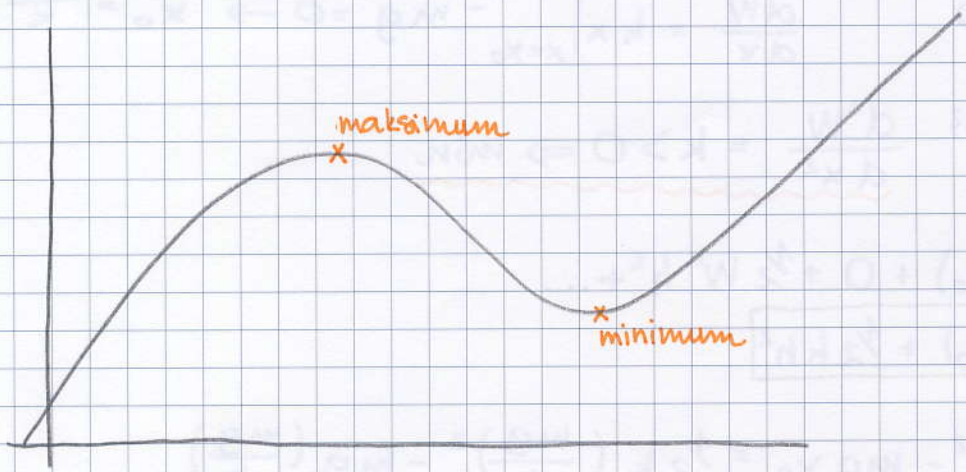
2. PRIMER

$$v = N_0 \cdot e^{-\beta t} \Big| \ln$$

$$\ln v = \ln(v_0) - \beta t \Big| \cdot d$$ – diferencial

$$\frac{dv}{v} = 0 - \beta dt$$

b) EKSTREMI

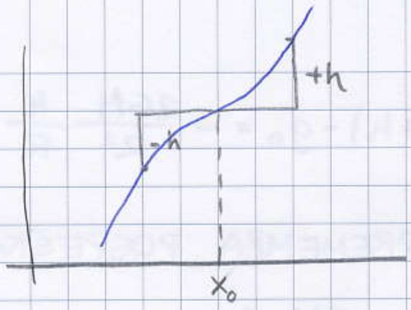


Ekstrem: $f'(x_0) = 0$
└ v točki x_0

$f(x_0 + h) = f(x_0) + \frac{1}{2} f''(x_0) h^2 + \ldots$ PRVEGA ODVODA NI, KER JE $\emptyset$!

MAKSIMUM $f''(x_0) < 0$
MINIMUM $f''(x_0) > 0$

$f''(x_0) = 0$

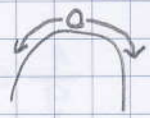


SEDLO, PREVOJ

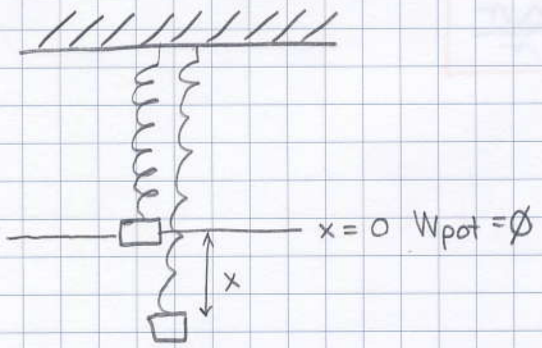
energija

$W(x)$ = minimum, RAVNOVESJE

$W'(x)=0 \rightarrow$ lahko je maksimum $\rightarrow$ to pa ni ravnovesje $\Rightarrow$ labilna lega



Telo na vzmeti



$$\left.\begin{aligned} W_{pr}(x) &= \tfrac{1}{2} k x^2 \\ W_{pot} &= -mgx \end{aligned}\right\} \text{ Skupna } W = \tfrac{1}{2} k x^2 - mgx$$

Ekstrem: $W' = 0$ $\qquad \dfrac{dW}{dx} = kx\Big|_{x=x_0} - mg = 0 \rightarrow x_0 = \dfrac{mg}{k}$

Minimum, maksimum? $\dfrac{d^2W}{dx^2} = k > 0 \Rightarrow \min$

$$W(x_0+h) = W(x_0) + 0 + \tfrac{1}{2} W'' h^2 + \ldots$$

$$\boxed{W(x_0+h) = W(x_0) + \tfrac{1}{2} k h^2}$$

$$W(x_0) = \tfrac{1}{2} k x_0^2 - mg x_0 = \tfrac{1}{2} k \left(\frac{mg}{k}\right)^2 - mg\left(\frac{mg}{k}\right)$$
$$= \frac{1}{2}\frac{m^2g^2}{k} - \frac{m^2g^2}{k} = -\frac{m^2g^2}{2k}$$

$W_{pr}(x_0) = -\tfrac{1}{2} W_{pot}(x_0)$

$W(x_0) = -W_{pr}(x_0) = \tfrac{1}{2} W_{pot}(x_0)$

$$W = a x^n + b x^m$$

$$W' = a n x_0^{n-1} + b m x_0^{m-1} = 0 \;\Big|\; \cdot x_0 \quad \Rightarrow x_0 = \ldots$$

$$a n x_0^n + b m x_0^m$$

$$W = W_a + W_b$$

$W_a = a x^n$ $\qquad$ $W_a(x_0) = a x_0^n$

$W_b = b x^m$ $\qquad$ $W_b(x_0) = b x_0^m$

$$a n x_0^n + b m x_0^m = 0$$

$$n W_a(x_0) + m W_b(x_0) = 0$$

$$n W_a(x_0) = - m W_b(x_0)$$

$$W(x_0) = W_a(x_0) + W_b(x_0)$$

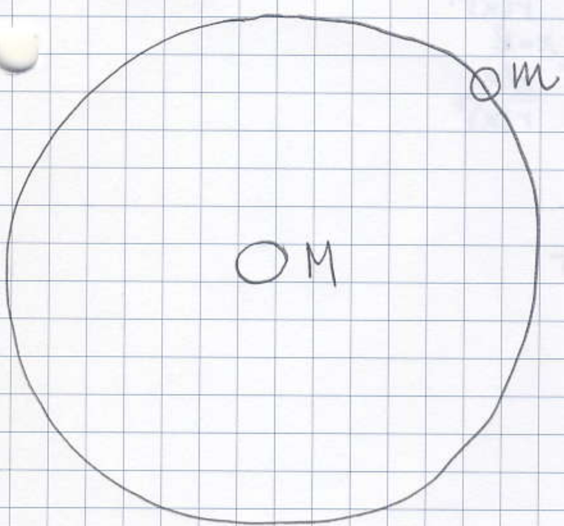
$$W(x_0) = W_a(x_0) - \frac{n}{m} W_a(x_0)$$

$$W(x_0) = \left(1 - \frac{n}{m}\right) W_a(x_0)$$

VIRIALNI TEOREM

$$W(x_0) = \frac{m-n}{m} W_a(x_0) \;\Big|\; n \neq m$$

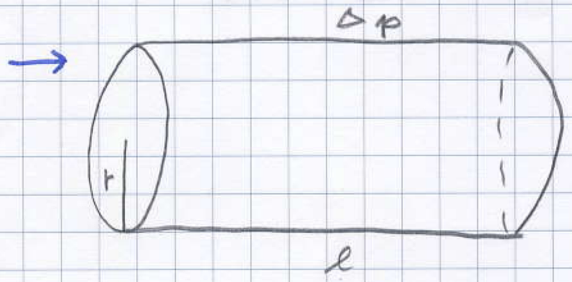
PRIMER:



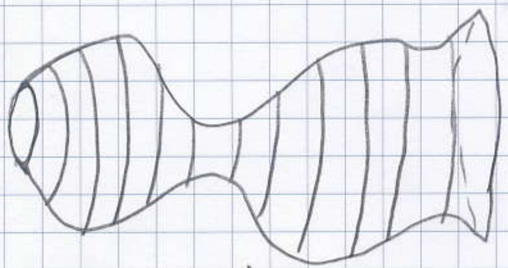
$W_k = \frac{1}{2} m v^2$ $\qquad$ $W_p = - \frac{G m M}{r}$

$$\Gamma = m r v \Rightarrow v = \frac{\Gamma}{m r}$$

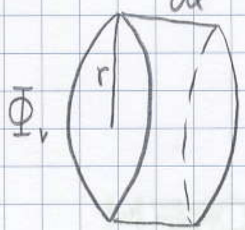
INTEGRALI v fiziki



$$\Phi_v = \frac{\pi}{8} \frac{r^4 \Delta p}{\eta l}$$



Skozi vsak kolobar teče isti volumski tok, kot skozi posamezen kolobar.



$$\Phi_v = \frac{\pi}{8} \frac{r(x)^4 \, dp}{\eta \, dx}$$

→ Prvo kar naredimo je to, da pogledamo, kaj se zelo malo spreminja.

Diferencial so zmeraj zgoraj v števcu.

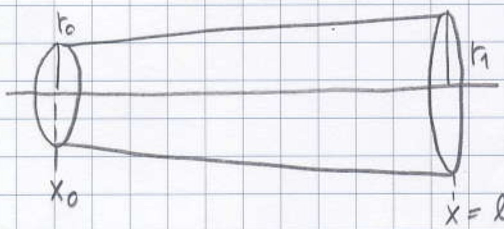
$$\Phi_v \frac{dx}{r(x)^4} = \frac{\pi}{8} \frac{dp}{\eta}$$

$$\int_{p_1}^{p_2} dp = \frac{8 \Phi_v \eta}{\pi} \frac{dx}{r(x)^4}$$

$$\Delta p = \frac{8 \eta \Phi_v}{\pi} \int_0^{x=l} \frac{dx}{r(x)^r}$$

$$\Phi_v = \frac{\pi \Delta p}{8 \eta} \frac{1}{\int_0^l \frac{dx}{r(x)^4}}$$

→ $r(x)$



$$r(x) = kx + n = r_0 + kx$$

$$r(l) = r_0 + kl = r_1 \Rightarrow k = \frac{r_1 - r_0}{l}$$

$$r(x) = r_0 + \left(\frac{r_1 - r_0}{l}\right) x$$

$$\int \frac{dx}{r(x)^4} = \int_0^l \frac{dx}{\left(r_0 + \frac{(r_1 - r_0)}{l} x\right)^4}$$

$$\int \frac{dx}{r(x)^4} = \int_0^\ell \frac{dx}{\left(r_0 + \frac{(r_1-r_0)}{\ell}x\right)^4}$$

$$\rightarrow u = r_0 + \frac{r_1-r_0}{\ell}x$$

$$du = \frac{r_1-r_0}{\ell}dx$$

$$\rightarrow dx = \frac{\ell}{r_1-r_0}du$$

$$\Rightarrow \int \frac{du}{u^4} \cdot \frac{\ell}{r_1-r_0} =$$

Trik (meje)

$$= \frac{\ell}{r_1-r_0}\int u^{-4}\,du =$$

$$= \frac{\ell}{r_1-r_0}\cdot\frac{1}{(-3)}\,u^{-3}\Big|_{x=0}^{x=\ell} =$$

$$= -\frac{1}{3(r_1-r_0)}\left(\frac{1}{u^3}\right)\Big|_{x=0}^{x=\ell} =$$

$$= -\frac{\ell}{3(r_1-r_0)}\left(\frac{1}{r_1^3}-\frac{1}{r_0^3}\right) =$$

$$= -\frac{\ell}{3(r_1-r_0)}\left(\frac{r_0^3-r_1^3}{r_1^3\cdot r_0^3}\right) =$$

$$= \frac{\cancel{(r_1-r_0)}(r_0^2+r_0r_1+r_1^2)}{3\cancel{(r_1-r_0)}(r_1^3\cdot r_0^3)} =$$

$$\Phi_V = \frac{\pi\,\Delta p}{8\,\eta}\,\frac{3\,r_1^3r_0^3}{(r_0^2+r_0r_1+r_1^2)\ell}$$

→ za preizkus lahko preverimo z $r_1 = r_0 = r$.

8.3.10

INTEGRACIJA ENAČB GIBANJA

1-DIMENZIJA

$$a = \frac{F}{m} = f(t, x, v)$$

$$\frac{dv}{dt} = f(t, x, v)$$ DIFERENCIALNA ENAČBA

Naša rešitev (do tu hočemo priti) $= v(t), x(t)$

① $\frac{dv}{dt} = f(t)$ Kako pridemo do integrala?

$$\int_{v_0}^{v} dv = \int^{t} f(t) \cdot dt$$

to je v pri času t_0

$t_0 = 0 \rightarrow$ da ni toliko za računat

ni nujno, da je v pri času t_0 tudi $\emptyset$

$$\boxed{v(t)} - v_0 = \int_0^t f(t)\, dt$$

$$\frac{dx}{dt} = v(t)$$

$$\int_{x_0}^{x} dx = \int_0^t v(t)\, dt$$

$$x(t) - x_0 = \int \left[v_0 + \int_0^t f(t')\, dt' \right] dt$$

$$\frac{dv}{dt} = f(v)$$

$$\int_{v_0}^{v} \frac{dv}{f(v)} = \int_0^t dt$$ VSE SPREMENLJIVKE NA ENO STRAN!

nova funkcija $\mathcal{F}(v) = t$

$$v(t) = \mathcal{F}^{-1}(t)$$ RECIPROČNA FUNKCIJA

$$x(t) = \int v(t)\, dt + x_0$$

$$\rightarrow \frac{dv}{dt} = f(v,t)$$

- $$\frac{dv}{dt} = kv\cdot t$$

$$\frac{dv}{v} = kt\,dt \quad ✓$$

- $$\frac{dv}{dt} = av + bt$$

$$\frac{dv}{dt} - av = bt \quad \Big|\, dt$$

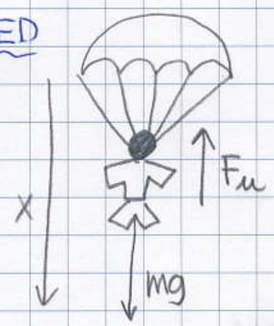
$$dv - av\,dt = bt\,dt$$

NE GRE!

SPLOŠNO: $$\frac{dv}{dt} = \underbrace{g(v)\cdot h(t)}_{\text{funkcij}}$$

$$\frac{dv}{g(v)} = h(t)g(t)$$

ZGLED



$$ma = mg - \tfrac{1}{2} c_u \rho S v^2$$

$$m\frac{dv}{dt} = mg - \tfrac{1}{2} c_u \rho S v^2$$

$$\frac{dv}{dt} = g - \boxed{\frac{1}{2m} c_u \rho S} v^2 \ !$$

$k \rightarrow$ izberemo si neko konstanto

$$\frac{dv}{dt} = \underbrace{g - kv^2}_{f(v)}; \quad k = \frac{c_u \rho S}{2m}$$

$$\int_{v_0}^{v} \frac{dv}{g - kv^2} = \int_0^t dt$$

UPORABA MATEMATIČNEGA PRIROČNIKA!

$$\int \frac{dx}{a^2 - x^2}$$

$$\int \frac{dv}{k\left(\underbrace{\frac{g}{k}}_{a^2} - v^2\right)} =$$

$$= \frac{1}{a}\,\mathrm{Arth}\,\frac{x}{a} = ① \ \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) \quad \text{za } a > x$$

Arth: aria hiperbolus tangens

$$② \ \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) \quad \text{za } x > a$$

$$\frac{1}{2ka} \ln \frac{x-a}{x+a}\Big|_{v_0}^{v} = \frac{1}{ka}\,\mathrm{Arth}\,\frac{x}{a}$$

$x = v$ (za vaš primer)

$$\frac{1}{ka}\,\mathrm{Arth}\,\frac{v}{a} = t$$

$$\mathrm{Arth}\,\frac{v}{a} = kat$$

$$\left(\mathrm{Arth}\frac{v}{a}\right)^{-1} = \mathrm{th}\frac{v}{a}$$

$$\frac{1}{2ka}\ln\left(\frac{v-a}{v+a}\right)\Big|_{v_0}^{v} = t$$

$$\frac{1}{2ka}\left[\ln\left(\frac{v-a}{v+a}\right) - \ln\left(\frac{v_0-a}{v_0+a}\right)\right] =$$

$$= \frac{1}{2ka}\ln\frac{(v-a)(v_0+a)}{(v+a)(v_0-a)}$$

$$\frac{(v-a)(v_0+a)}{(v+a)(v_0-a)} = e^{2akt}$$

$$a^2 = \frac{g}{k}$$

$$\frac{dv}{dt} = g - kv^2$$

$v = \text{konst} = v_1$ $\qquad \frac{dv}{dt} = \emptyset \qquad g = kv_1^2$

$$v_1^2 = \frac{g}{k}$$

$\hookrightarrow$ končna hitrost

$$\frac{dv}{dt} = f(x) \qquad x = \int v\,dt$$

$$\frac{dx}{dt} = v$$

DIFERENCIALNI SISTEM ENAČB

Nekaj moramo eliminirati:

$$dt = \frac{dx}{v} \rightarrow \text{druga enačba}$$

$$\frac{dv}{\frac{dx}{v}} = f(x)$$

$$v\cdot\frac{dv}{dx} = f(x) \quad \Big|\cdot dx$$

$$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} f(x)\cdot dx$$

$$\frac{1}{2}v^2\Big|_{v_0}^{v} = \int_{x_0}^{x} f(x)\,dx$$

$$\int f\,dx = \mathcal{J}(x)$$

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \int_{x_0}^{x} f(x)\,dx = \mathcal{J}(x) - \mathcal{J}(x_0)$$

$$\frac{1}{2}v^2 = \frac{1}{2}v_0^2 + \mathcal{J}(x) - \mathcal{J}(x_0) \quad \Big|\cdot 2$$

$$v^2 = v_0^2 + 2(\mathcal{J}(x) - \mathcal{J}(x_0))$$

$$v = \pm\sqrt{v_0^2 + 2(\mathcal{Y}(x) - \mathcal{Y}(x_0))}$$

$\pm$ → izberemo tistega, ki nam da smiselno rešitev

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \int_{x_0}^{x} f(x)\,dx = \mathcal{Y}(x) - \mathcal{Y}(x_0) \quad \Big| \cdot m$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \underbrace{\int F(x)\,dx}_{\text{delo}} = \underbrace{m\mathcal{Y}(x) - m\mathcal{Y}(x_0)}_{-W_{pot}(x) + W_{pot}(x_0)}$$

$$\frac{dx}{dt} = \pm\sqrt{v_0^2 + 2(\mathcal{Y}(x) - \mathcal{Y}(x_0))}$$

$$\int_{x_0}^{x} \frac{dx}{\sqrt{v_0^2 + 2(\mathcal{Y}(x) - \mathcal{Y}(x_0))}} = \pm\int_0^t dt$$

$$G(x) = \pm t$$

$$x(t) = G^{-1}(\pm t)$$

$$v(t) = \frac{dx}{dt}$$

→ $F = -kx$

$$f = \frac{F}{m} = -\frac{k}{m}x$$

① Izračunaj $\mathcal{Y}(x)$

$$\mathcal{Y}(x) = \int f(x)\,dx = -\frac{k}{m}\frac{x^2}{2}$$

② $v(x) = \pm\sqrt{v_0^2 + 2(\mathcal{Y}(x) - \mathcal{Y}(x_0))}$

③ $x_0 = s_0$, $\quad v_0 = 0$ → ker jo spustimo

$$v(x) = \pm\sqrt{2(\mathcal{Y}(x) - \mathcal{Y}(x_0))}$$

Noter vstavimo: $\mathcal{Y}(x) = -\frac{k}{m}\frac{x^2}{2}$

$$v(x) = \pm\sqrt{\frac{k}{m}(s_0^2 - x^2)}$$

2 se pokrajšata

$$\frac{dx}{dt} = \pm\sqrt{v_0^2 + 2(\mathcal{Y}(x) - \mathcal{Y}(x_0))}$$

Akta

$$\int_{s_0}^{x} \frac{dx}{\sqrt{\frac{k}{m}(s_0^2 - x^2)}} = \pm \int_0^t dt$$

$$\sqrt{\frac{m}{k}} \int_{s_0}^{x} \frac{dx}{(s_0^2 - x^2)} = \pm dt$$

$$\sqrt{\frac{m}{k}} \int_{s_0}^{s} \frac{dx}{\sqrt{s_0^2 - x^2}} = \pm dt$$

$$\sqrt{\frac{m}{k}} \arcsin \frac{x}{s_0} \Big|_{s_0}^{x} = \pm t$$

$$\sqrt{\frac{m}{k}} \left(\arcsin \frac{x}{s_0} - \underbrace{\arcsin 1}_{\pi/2}\right) = \pm 1$$

$$\arcsin \frac{x}{s_0} - \frac{\pi}{2} = \pm \sqrt{\frac{k}{m}}\, t$$

$$\arcsin \frac{x}{s_0} = \pm \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}$$

$$\frac{x}{s_0} = \sin\left(\pm \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right)$$

MATEMATIČNI PRIR.

$\sqrt{a^2 - x^2}$ $\qquad x = a^2 - x^2$

$$\int \frac{dx}{\sqrt{x}} = \arcsin \frac{x}{a}$$

Rešitev le če $x^2 \leq a^2$.

$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$

$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$

$$x = s_0 \cos \sqrt{\frac{k}{m}}\, t$$

$$v = \frac{dx}{dt} =$$

$$= -s_0 \sin \sqrt{\frac{k}{m}} \cdot t \cdot \sqrt{\frac{k}{m}}$$

$$= -s_0 \sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

VEKTORJI

16.3.10

SKALARJI – pomembna je le velikost
– iz $\mathbb{R}$
– čas, masa, delo, ...

} seštevanje, množenje

VEKTORJI – hitrost, sila
– zanima velikost in smer
– P – prostor za vektorje

$\mathbb{R} \to \mathbb{R} \qquad a \mapsto a$

$\mathbb{R}, \mathbb{R} \to \mathbb{R} \qquad (a,b) \mapsto a+b$

seštevanje

$P \to P \qquad \vec{a} \mapsto -\vec{a}$

$P, P \to P \qquad (\vec{a}, \vec{b}) \mapsto \vec{a} + \vec{b}$

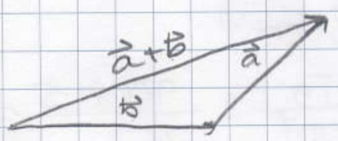
Obstaja 0 $\qquad \vec{a} + 0 = \emptyset + \vec{a} = \vec{a}$

Nasprotna operacija:

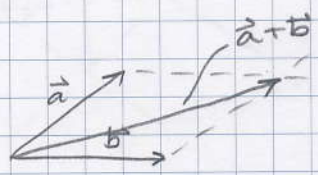
$\vec{a} + \vec{b} = \emptyset \; ; \qquad \vec{a} = -\vec{b}$

Vektorje predstavimo kot usmerjene daljice.

① TRIKOTNIŠKO PRAVILO

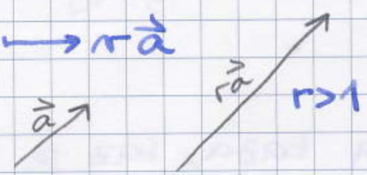


② PARALELOGRAMSKO PRAVILO



$\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$

$(\mathbb{R}, P) \to P \qquad (r, \vec{a}) \longmapsto r\vec{a}$

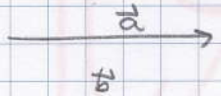


$r(s\vec{a}) = (rs)\vec{a}$ ASOCIATIVNO MNOŽENJE

LINEARNA KOMBINACIJA

$r_1\vec{a}_1 + r_2\vec{a}_2 + r_3\vec{a}_3 + \dots$

(NE) ODVISNOST



$\Rightarrow$ 2 možnosti: $\vec{b} \parallel \vec{a}$; $\vec{b} = r\vec{a}$

$\vec{b} \nparallel \vec{a}$; $r\vec{a} + s\vec{b} = 0$

$\parallel$

$r\vec{a} + s(x\vec{a}) = 0$

$(r + sx)\vec{a} = 0$

$s = -\frac{r}{x}$

$\nparallel$

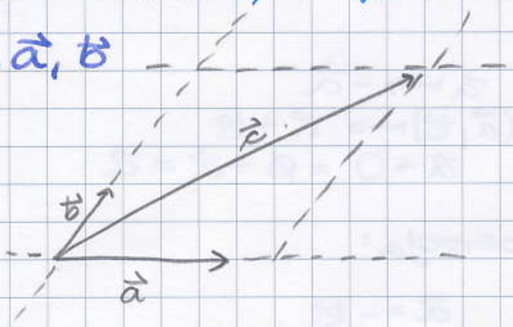
$r = 0$

$s = 0$

TRIVIALNA REŠITEV

$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n, \dots$ so neodvisni, takrat ko velja $r_1\vec{a}_1 + r_2\vec{a}_2 + \dots + r_n\vec{a}_n = 0$ natanko takrat, ko je $r_1 = r_2 = \dots = r_n = 0$.

$\vec{c}, \vec{a}, \vec{b}$



$\vec{c} = r\vec{a} + s\vec{b}$

$\vec{a}, \vec{b}$ sta baza vektorskega prostora, ker vsak drug vektor lahko predstavimo kot linearno kombinacijo.

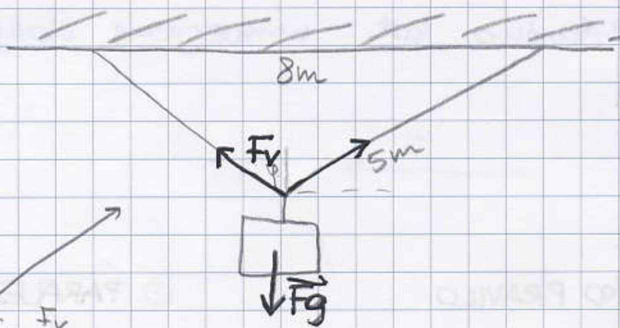
ZGLED

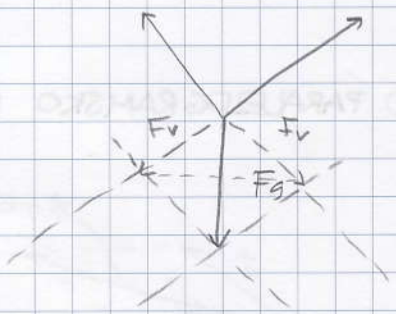
vrv ... 10m

utež ... $F_g = 900N$

$l = 8m$

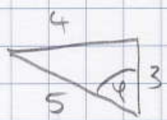
$\vec{F_v} = ?$





$F_g = 2 \cdot F_v \cdot \cos\varphi$

$F_v = \frac{F_g}{2\cos\varphi} = \frac{900N \cdot 5}{6} \approx 750N$



$\vec{c} = \underbrace{r\vec{a} + s\vec{b}}$

vsta le onadva baza, kaz je več, npr. $\vec{a}', \vec{b}'$

$\vec{a}, \vec{b}$ = baza $\Rightarrow$ LINEARNO NEODVISNA

$\vec{d} = r_2\vec{a} + s_2\vec{b}$ $\vec{c} = r_1\vec{a} + s_1\vec{b}$

$\vec{d} = (r_2, s_2) = \begin{Vmatrix} r_2 \\ s_2 \end{Vmatrix}$ $\vec{c} = (r_1, s_1) = \begin{Vmatrix} r_1 \\ s_1 \end{Vmatrix}$ koordinate

$\boxed{\vec{c} + \vec{d}} = (r_1\vec{a} + s_1\vec{b}) + (r_2\vec{a} + s_2\vec{b}) =$

$= \vec{a}(r_1 + r_2) + \vec{b}(s_1 + s_2) = r\vec{a} + s\vec{b}$

seštevanje po komponentah

$\begin{Vmatrix} r_1 \\ s_1 \end{Vmatrix} + \begin{Vmatrix} r_2 \\ s_2 \end{Vmatrix} = \begin{Vmatrix} r_1 + r_2 \\ s_1 + s_2 \end{Vmatrix}$

SKALARNI PRODUKT

$\underbrace{P, P}_{P^2} \to \underset{\text{skalar}}{\mathbb{R}}$ $(\vec{a}, \vec{b}) \longmapsto \vec{a} \cdot \vec{b}$

LASTNOSTI:

① KOMUTATIVNOST $\vec{a}\vec{b} = \vec{b}\vec{a}$

② ASOCIATIVNOST $(r\vec{a})\vec{b} = r(\vec{a}\vec{b})$

③ DISTRIBUTIVNOST $\vec{a}(\vec{b} + \vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}$

$(\vec{a} \cdot \vec{b})\vec{c} \cancel{=} \vec{a}(\vec{b} \cdot \vec{c})$ NI MOŽNO!

④ $\left.\begin{matrix} \vec{a} \cdot \vec{b} = 0 \\ \vec{a} \neq 0 \\ \vec{b} \neq 0 \end{matrix}\right\} \vec{a} \perp \vec{b}$

⑤ $\vec{a} \cdot \vec{a} \geq \emptyset$ POZITIVNA DEFINITNOST

$\vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{a} = 0$

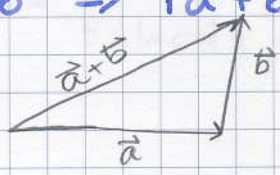
DOLŽINA vektorja

$d = \sqrt{\vec{a} \cdot \vec{a}} = a$ $d(\vec{a}) = |\vec{a}|$

$d(\vec{a} + \vec{b}) = \sqrt{(\vec{a} + \vec{b})(\vec{a} + \vec{b})} =$

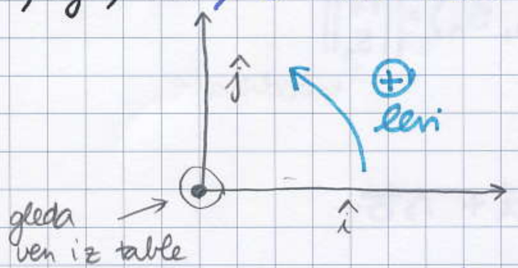
$= \sqrt{a^2 + 2\vec{a}\vec{b} + b^2}$

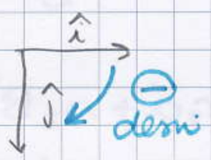
$\vec{a} \perp \vec{b} \Rightarrow |\vec{a} + \vec{b}|^2 = a^2 + b^2 = |\vec{a}|^2 + |\vec{b}|^2$



KARTEZIČNI KOORDINATNI SISTEM

$\vec{i}, \vec{j}, \vec{k}$, ki so med seboj pravokotni.





$|\vec{i}| = 1$ $\quad \hat{i}$

DOKAZ, da ni drugih možnosti

$\hat{j}\cdot\hat{i} = 0 \qquad \hat{j}\cdot\hat{j} = 1$

$\hat{i}\cdot\hat{i} = 1$

$\vec{n} = r\hat{i} + s\hat{j}$

$n\hat{i} = \emptyset \qquad \vec{n}\vec{n} = 1$

$(r\hat{i} + s\hat{j})\hat{i} = r = \emptyset \Rightarrow n = s\hat{j}$

$s^2\hat{j}\hat{j} = 1$

$s^2 = 1$

$s = 1 \quad s = -1$

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\hat{i}\hat{i} = \hat{j}\hat{j} = \hat{k}\hat{k} = 1$

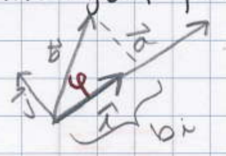
$\hat{i}\hat{j} = \hat{i}\hat{k} = \hat{j}\hat{k} = \emptyset$

$\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\vec{a}\cdot\vec{b} = x_1x_2 + y_1y_2 + z_1z_2$

$\left\|\begin{matrix}x_1\\y_1\\z_1\end{matrix}\right\| \cdot \left\|\begin{matrix}x_2\\y_2\\z_2\end{matrix}\right\| = x_1x_2 + y_1y_2 + z_1z_2$

$\vec{a}\cdot\vec{b} \rightarrow$ geometrijski pomen



$b_a = b\cos\varphi$

$\vec{a} = a\hat{i}$

$\underbrace{\vec{b} = b_i\hat{i} + b_j\hat{j}}$

$\vec{a}\cdot\vec{b} = a\hat{i}\,(b_i\hat{i} + b_j\hat{j}) = a\,b_i =$

dolžina a

dolžina b v smeri $\hat{i}$

$= a\cdot b_a =$ projekcija b na a

$= a\,b\cos\varphi = b\,a_b$

UPORABA:

→ $|\vec{a}\cdot\vec{b}| \leq \vec{a}\cdot\vec{b}$ TRIKOTNIŠKA NEENAKOST – ker je $|\cos\alpha| \leq 1$

→ $\cos\varphi = \dfrac{\vec{a}\,\vec{b}}{a\,b}$

$\rightarrow |\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$

$$\sqrt{a^2+2\vec{a}\vec{b}+b^2}$$

$$a^2-2\vec{a}\vec{b}+b^2 = (a-b)^2$$

$$||a|-|b|| \leq |\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$$

ZGLED

$(12, 13, 20, 4)$ ... cene $= \vec{c}$

$(1, 2, 4, 10)$ ... količine $= \vec{k}$

Koliko moramo plačati?

$\$ = \vec{k} \cdot \vec{c}$

$\leq |\vec{k}||\vec{c}|$

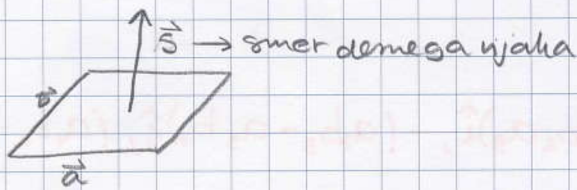
$c' = (4, 12, 13, 20) \rightarrow$ da bi čim več plačal

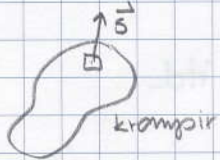
PRIMERI V FIZIKI:

$\rightarrow A = \vec{F} \cdot \vec{s}$

$\rightarrow \Phi_v = \vec{S}\vec{v}$

└ ploskev





VEKTORSKI PRODUKT

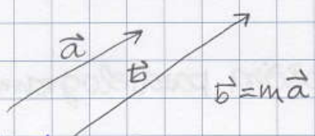
23. 3. '10

Definiran le v $\mathbb{R}^3$. $\quad \mathcal{E}^2_\varepsilon \rightarrow \mathcal{E}_3$

$(\vec{a}, \vec{b}) \longrightarrow \vec{a} \times \vec{b}$

LASTNOSTI:

1. $\vec{a} \times \vec{b} \rightarrow \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$
2. $\vec{a} \times \vec{a} = -\vec{a} \times \vec{a} = 0$
3. $(m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b})$
4. Če sta vektorja vzporedna $\rightarrow \vec{a} \parallel \vec{b} \quad \vec{a} \times \vec{b} = 0$



5. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

LEVO SUČNI TRIROB (POZITIVNI)

Pozitivna baza $\hat{i}, \hat{j}, \hat{k}$

$\hat{i} \times \hat{j} = \hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$
$$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\vec{a} \times \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) =$$

$$= a_1 b_2 \hat{k} + a_1 b_3 (-\hat{j}) + a_2 b_1 (-\hat{k}) + a_2 b_3 \hat{i} + a_3 a_1 \hat{j} + a_3 b_2 (-\hat{i}) =$$ ANTIKOMUTATIVNOST

$$= (a_2 b_3 - a_3 b_2)\hat{i}$$
$$+ (a_3 b_1 - a_1 b_3)\hat{j}$$
$$+ (a_1 b_2 - a_2 b_1)\hat{k}$$

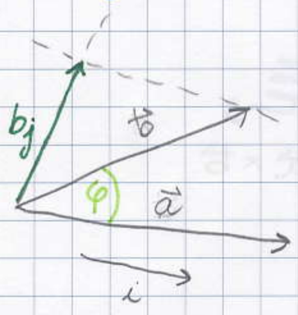
V vsakem členu nastopajo vse tri številke, vsaka po enkrat.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - b_2 a_3)\hat{i}, -(a_1 b_3 - a_3 b_1)\hat{j}, (a_1 b_2 - a_2 b_1)\hat{k}$$

PRI DETERMINANTI PAZI NA PREDZNAKE $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ itd.

↳ to gledaš glede na menjave

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

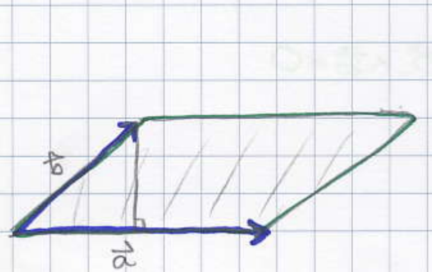


$$b_j = b \sin\varphi$$

$$\vec{a} = a\hat{i}$$
$$\vec{b} = b_i \hat{i} + b_j \hat{j}$$

$$\vec{a} \times \vec{b} = a\hat{i} \times (b_i \hat{i} + b_j \hat{j}) =$$
$$= a \underbrace{b_j}_{b \sin\varphi} \hat{k}$$

$$|\vec{a} \times \vec{b}| = a \cdot b \cdot \sin\varphi$$



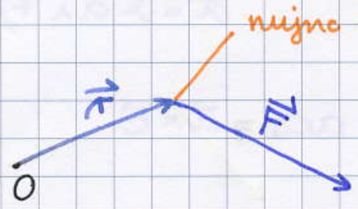
$\vec{a} \times \vec{b}$ = ploščina paralelograma

$$|\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = a^2 b^2$$ Lagrangejeva identiteta

$\vec{a}\times\vec{b} \to \perp \vec{a}$

$\quad\quad\quad \to \perp \vec{b}$

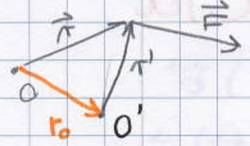
$$\vec{M} = \vec{r}\times\vec{F}$$



Če imamo ravnovesje sil in navorov, je vseeno okoli katere osi računamo.
če velja dobimo 0, potem okoli vsake osi 0.

$$\sum_i \vec{F}_i = 0$$
$$\sum_i \vec{M}_i = 0$$

Poglejmo, če to drži v nekem drugem izhodišču



$$\vec{r}' = -\vec{r}_0 + \vec{r}$$

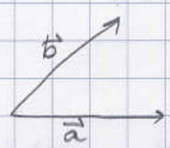
$$\sum_i \vec{M}_i' = \sum \vec{r}' \times \vec{F}_i =$$
$$= \sum_i (\vec{r}_i - \vec{r}_0)\times \vec{F}_i =$$
$$= \sum_i \left(\vec{r}_i \times \vec{F}_i - \vec{r}_0 \times F_i\right) =$$
$$= \underbrace{\sum_i \vec{r}_i \times \vec{F}_i}_{0} - \sum_i \vec{r}_0 \times \vec{F}_i =$$
$$= -\vec{r}_0 \times \underbrace{\sum_i F_i}_{0} = \emptyset$$

konstopa ✓ vseh

DVOJNI VEKTORSKI PRODUKT

$$(\vec{a}\times\vec{b})\times\vec{c}$$

$$\vec{d} = (\vec{a}\times\vec{b})\times\vec{c} \longrightarrow \begin{matrix}\vec{d} \perp \vec{a}\times\vec{b} \\ \vec{d} = \lambda\vec{a} + \mu\vec{b}\end{matrix}$$



$\vec{a} = a\hat{i}$

$\vec{b} = b_i\hat{i} + b_j\hat{j}$

$\vec{c} = c_i\hat{i} + c_j\hat{j} + c_k\hat{k}$

$$\vec{a}\times\vec{b} = a b_j \hat{k}$$

$$(a b_j \hat{k})\times(c_i\hat{i} + c_j\hat{j} + c_k\hat{k}) =$$

$$ab_j\hat{k}\times(c_i\hat{\imath}+c_j\hat{\jmath}+c_k\hat{k}) = ab_jc_i\hat{\jmath} - ab_jc_j\hat{\imath} = \vec{d}$$

Po drugi strani pa zapišemo $\vec{d} = \lambda\vec{a} + \mu\vec{b}$.

$$\vec{d} = \lambda a\hat{\imath} + \mu b_i\hat{\imath} + \mu b_j\hat{\jmath}$$

Če primerjamo po komponentah:

$$\hat{\jmath}:\ ab_jc_i = \mu b_j \iff \mu = ac_i = \vec{a}\times\vec{c}$$

$$\hat{\imath}:\ -ab_jc_j = \lambda a + \mu b_i$$

$$-\cancel{a}b_jc_j = \lambda\cancel{a} + \cancel{a}c_ib_i$$

$$\lambda = -(b_ic_i + b_jc_j) =$$

$$= -(\vec{b}\vec{c})$$

$c_i = c_A$ c_i je projekcija na $\vec{a}$

$c_A a = \vec{a}\cdot\vec{c}$

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$$

$$(\vec{b}\times\vec{c})\times\vec{a} = (\vec{b}\vec{a})\vec{c} - (\vec{c}\vec{a})\vec{b}$$

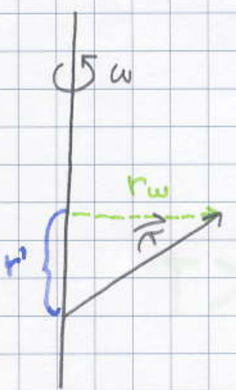
$$(\vec{c}\times\vec{a})\times\vec{b} = (\vec{c}\vec{b})\vec{a} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$\Sigma = 0$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\vec{c})\vec{b} - (\vec{a}\vec{b})\vec{c}$$

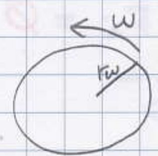
NI ISTO!

$$(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$$



$$a_r = \omega^2 r_\omega$$

$$\vec{a}_r = -\omega^2 r_\omega \cdot \frac{\vec{r}_\omega}{r_\omega}$$



$$v = \omega r_\omega$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \vec{\omega}\times\vec{v}$$

$$\vec{\omega}\times\vec{r} = \vec{v}$$

$$\vec{\omega} = \omega\frac{\vec{\omega}}{|\vec{\omega}|}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \vec{\omega}(\underbrace{\vec{\omega}\vec{r}}_{\omega r'}) - \vec{r}(\omega^2) =$$

$$= \omega^2 r'\frac{\vec{\omega}}{\omega} - \vec{r}\omega^2 =$$

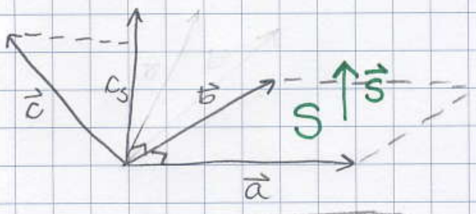
$$= \omega^2\Big(-\vec{r} + \underbrace{r'\frac{\vec{\omega}}{\omega}}_{\vec{r}}\Big) = \omega^2\vec{r}_\omega$$

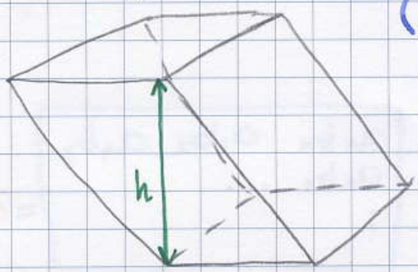
$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega^2 \vec{r}_\omega = -\vec{a}_r$$

$$\boxed{\vec{a}_r = (\vec{\omega}\times\vec{r})\times\vec{\omega}}$$

MEŠANI PRODUKT

$$(\vec{a}\times\vec{b})\cdot\vec{c}$$





$$\vec{a}\times\vec{b} = \vec{S}$$

$$(\vec{a}\times\vec{b})\vec{c} = \vec{S}\vec{c} = \vec{S}c_s = V_{\vec{a},\vec{b},\vec{c}}$$

$$= V_{\vec{b}\vec{c}\vec{a}}$$

$(\vec{a}\times\vec{b})\vec{c} = (\vec{b}\times\vec{c})\vec{a} = (\vec{c}\times\vec{a})\vec{b}$ – velja če je pozitiven trirob

30.3.'10

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underset{\hat e_1}{\underset{\parallel}{\hat\imath}}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \underset{\hat e_2}{\underset{\parallel}{\hat\jmath}}\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + \underset{\hat e_3}{\underset{\parallel}{\hat k}}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) \qquad \vec{a} = \hat\imath a_1 + \hat\jmath a_2 + \hat k a_3 = \sum_i \hat e_i a_i$$

$$\vec{a}(\vec{b}\times\vec{c}) = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

Vsaka menjava vrstice spremeni predznak.

$$\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} \vec{b} \\ \vec{c} \\ \vec{a} \end{vmatrix}$$

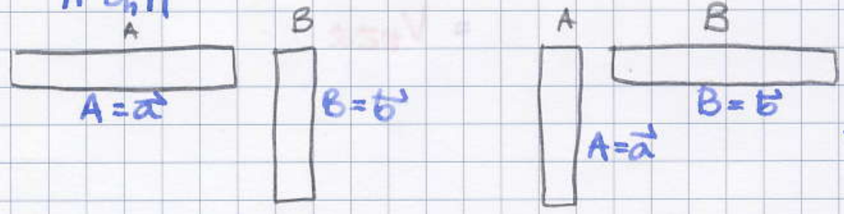
└ tu smo 2× spremenili vrstico, zato je spet +

TENZORSKI PRODUKT

$$\mathcal{E}_3^2 \to \mathcal{E}^2$$
$$(\vec{a}, \vec{b}) \mapsto \vec{a} \otimes \vec{b}$$

PONOVITEV MATRIK
$$C = A \cdot B = \sum a_{ij} b_{jk} = c_{ik}$$

$$\vec{a} = (a_1, a_2, \dots, a_n)$$
$$\vec{b} = \begin{Vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Vmatrix}$$



$$= \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & \dots & \\ \vdots & & \end{bmatrix} = \vec{a} \otimes \vec{b}$$

tenzorsko

$$c_{11} = \vec{a} \cdot \vec{b}$$

$$\vec{a} = (a_1, a_2, \dots, a_n) \quad \text{kot matrika} \quad \vec{a} = \begin{Vmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Vmatrix}$$

$A \dots a_{ij}$

$A^T \dots a_{ji}$ transponirano

$$\begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & \dots & \\ \vdots & & \end{bmatrix} =^T \begin{bmatrix} a_{11} & a_{21} & \dots \\ a_{12} & \dots & \\ \vdots & & \end{bmatrix}$$

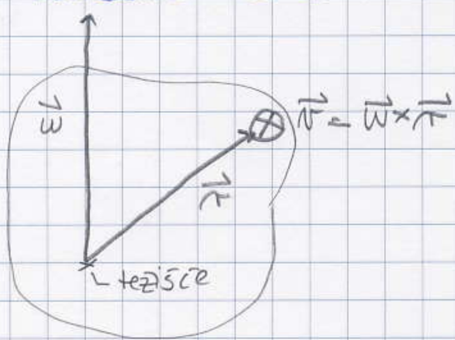
$$\vec{a}^T = \| a_1, a_2, \dots, a_n \|$$
$$\vec{a} \cdot \vec{b} = \vec{a}^T \cdot \vec{b}$$
$$\vec{a} \otimes \vec{b} = \vec{a} \cdot \vec{b}^T$$

$$\underbrace{\vec{a} \otimes \vec{b}}_{\text{matrika } A} \neq \underbrace{\vec{b} \otimes \vec{a}}_{\text{matrika } A^T}$$

PRIMER

$\vec{\Gamma} = J \vec{\omega}$ (vendar Γ in ω nimata vedno iste smeri $\Rightarrow$ tenzorji)

$$\vec{\Gamma} = \vec{r} \times \vec{G}$$
$$d\vec{\Gamma} = \vec{r} \times \vec{v}\, dm$$
$$\vec{\Gamma} = \int \vec{r} \times \vec{v}\, dm$$



$$\vec{\Gamma} = \int \vec{r} \times (\overset{+ \;\to\; \text{ker je v vrečini}}{\vec{\omega}} \times \vec{r})\, dm$$

$$\vec{\Gamma} = \int [\vec{\omega}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{\omega}\cdot\vec{r})]\, dm$$

→ Produkt matrike in vektorja

$$A_{nn} \cdot \vec{b}_n = \underbrace{\boxed{\quad}}_{n \times n} \cdot \underbrace{\boxed{\;}}_{n \times 1} = \underbrace{\boxed{\;}}_{n \times 1}$$

$$A \cdot \vec{b} = \vec{c} \qquad c_i = \sum_j a_{ij} b_j$$

$$\vec{\omega}\cdot\vec{r} = \omega_1 r_1 + \omega_2 r_2 + \omega_3 r_3$$

kot a

$$\vec{r}\cdot(\vec{\omega}\cdot\vec{r}) = \begin{Vmatrix} r_1 r_1 \omega_1 + r_1 r_2 \omega_2 + r_1 r_3 \omega_3 \\ r_2 r_1 \omega_1 + r_2 r_2 \omega_2 + r_2 r_3 \omega_3 \\ r_3 r_1 \omega_1 + r_3 r_2 \omega_2 + r_3 r_3 \omega_3 \end{Vmatrix} = \Rightarrow$$ to hočemo zapisati v obliki $A\cdot\vec{\omega}$

$$a_{ij}\cdot \omega_j = [\vec{r}(\vec{\omega}\vec{r})]_i$$

$$= \underbrace{\begin{Vmatrix} r_1 r_1 & r_1 r_2 & r_1 r_3 \\ r_2 r_1 & r_2 r_2 & r_2 r_3 \\ r_3 r_1 & r_3 r_2 & r_3 r_3 \end{Vmatrix}}_{= \vec{r}\otimes\vec{r}} \cdot \begin{Vmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Vmatrix} = (\vec{r}\otimes\vec{r})\cdot\vec{\omega}$$

to je matrika

$$\Gamma = \int [\vec{\omega}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{\omega}\cdot\vec{r})]\, dm =$$

$$= \int [r^2\vec{\omega} - (\vec{r}\otimes\vec{r})\vec{\omega}]\, dm =$$

$$= \left[\int (r^2 - \underbrace{\vec{r}\otimes\vec{r}}_{\text{matrika}})\, dm\right]\vec{\omega}$$

$$\to (A\cdot\vec{b})_i = \sum_j a_{ij} b_j$$

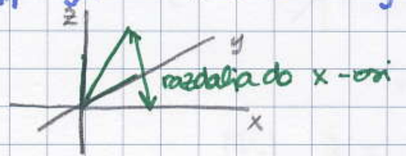
KRONECKERJEV SIMBOL $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ — IDENTITETA

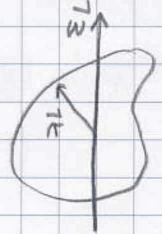
$$(s\cdot\vec{b})_i = s\cdot \vec{b}_i = s\sum_j \delta_{ij}\cdot b_j = \sum_j (s\,\delta_{ij} b_j)$$

$$b_i = \sum_j \cdot \delta_{ij}\, b_j$$

$$\to \vec{\Gamma} = \int (r^2 - \vec{r}\otimes\vec{r})\, dm \cdot \vec{\omega} = \underbrace{\int (r^2 I - \vec{r}\otimes\vec{r})\, dm}_{J} \cdot \vec{\omega}$$

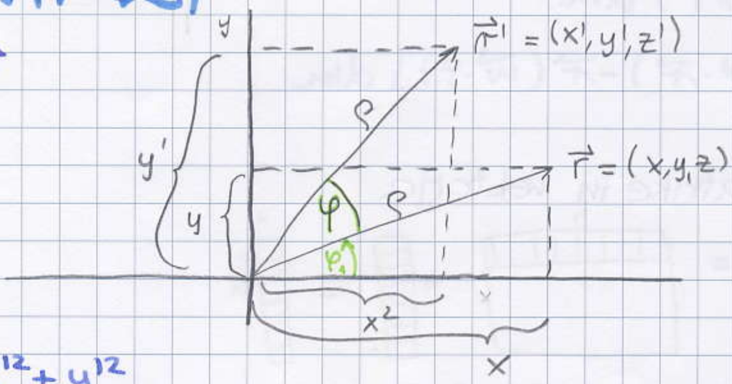
$$J_{11} = \int (r^2 \overset{I=1}{-} r_x^2)\, dm = \int (\cancel{r_x^2} + r_y^2 + r_z^2 - \cancel{r_x^2})\, dm = \int (r_y^2 + r_z^2)\, dm = \int (y^2 + z^2)\, dm$$





VRTEŽI

$$\vec{r}\,' = R(\varphi)\vec{r}$$



$$z' = z$$

$$\rho^2 = x^2 + y^2 = x'^2 + y'^2$$

$$x = \rho\cos\varphi_1$$
$$y = \rho\sin\varphi_1$$

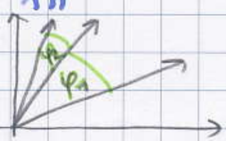
$$\begin{cases} x' = \rho\cos(\varphi+\varphi_1) = \rho(\cos\varphi\cos\varphi_1 - \sin\varphi\sin\varphi_1) \\ y' = \rho\sin(\varphi+\varphi_1) = \rho(\sin\varphi\cos\varphi_1 + \cos\varphi\sin\varphi_1) \end{cases}$$

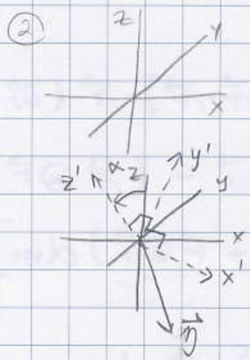
$$x' = x\cos\varphi - y\sin\varphi$$
$$y' = x\sin\varphi + y\cos\varphi$$
$$z' = z$$

$$\begin{Vmatrix} x' \\ y' \\ z' \end{Vmatrix} = \underbrace{\begin{Vmatrix} \cos\varphi & \sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix}}_{} \begin{Vmatrix} x \\ y \\ z \end{Vmatrix}$$

$R_z(\varphi)$ ... rotacija okoli z osi za kot φ

$$R_z(-\varphi) = \begin{Vmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix} = R_z^T(\varphi)$$





Kako zasukati vektor v poljubno zasukan k.o. ?

$$\vec{\eta} = \hat{z} \times \hat{z}'$$

$$\underbrace{\begin{aligned} \vec{r}\,' &= R_z(\varphi_1)\vec{r} \\ \vec{r}\,'' &= R_z(\varphi_2)\vec{r}\,' \end{aligned}}_{}$$

$$\vec{r}\,'' = \underbrace{R_z(\varphi_2)R_z(\varphi_1)}_{}\vec{r}$$

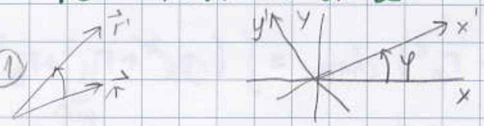
$$R_z(\varphi_2+\varphi_1) = \begin{Vmatrix} \cos(\varphi_1+\varphi_2) & \cdots \\ \vdots & \ddots \end{Vmatrix}$$

$$R_z(-\varphi)R_z(\varphi) = I$$

$$\boxed{R_z^{-1}(\varphi) = R_z(-\varphi) = R_z^T(\varphi)}$$

$R^{-1} = R^T$ ... unitarne

$$R_x(\varphi) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{Vmatrix}$$



$S: \vec{r}$
$S' = \vec{r}\,'$

$$\vec{r}\,' = U_z(\varphi)\vec{r}$$

$$U_z(\varphi) = R_z(-\varphi)$$

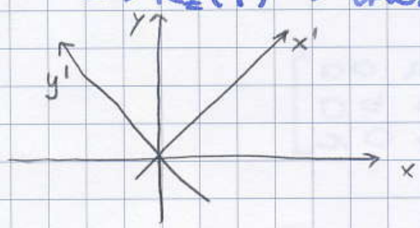
$$R_y(\varphi) = \begin{Vmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{Vmatrix}$$

Transformacije

6.4.10

$\vec{r} \to R\vec{r}$

$\hookrightarrow R_z(\varphi) \to$ uteži



$S \xrightarrow[\varphi]{} S'$

$\vec{r}\,' = U\vec{r}$, $U = R(-\varphi)$

$\underbrace{\vec{\Gamma} = J\vec{\omega}}_{S \to S'}$

$U\vec{\Gamma} = UJ\vec{\omega}$

$\vec{\Gamma}' = J'\vec{\omega}'$

$U\vec{\Gamma}' = UJU^{-1}(U\vec{\omega})$

$\vec{\omega}' = U\vec{\omega}$

$\hookrightarrow$ prehodna matrika

$J' = UJU' \longrightarrow$ kako se transformirajo tenzorji

Matrike

$\hookrightarrow$ tenzor (vztrajnostni)

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$\vec{\Gamma} \nparallel \vec{\omega}$ (ker pomnožimo s tenzorjem)

$$\vec{\Gamma} = J\vec{\omega} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix}$$

$\vec{\Gamma} = \lambda\vec{\omega}$

$\Gamma_1 = \lambda\omega_1$
$\Gamma_2 = \lambda\omega_2$
$\Gamma_3 = \lambda\omega_3$

$$J_0 = \begin{bmatrix} J_{11} & & \\ & J_{22} & \\ & & J_{33} \end{bmatrix}$$ $\longrightarrow$ če je tako, potem sta si Γ in ω vzporedna

$J' = UJ_0U' \quad |\cdot U \to$ z desne

$J'U = UJ_0\underbrace{U'U}_{I} \quad |U^{-1}\cdot \to$ z leve

$U^{-1}J'U = J_0$

$$J_0\vec{\omega} = \begin{bmatrix} J_{11}\omega_1 \\ J_{22}\omega_2 \\ J_{33}\omega_3 \end{bmatrix}$$

$\vec{w} = \begin{vmatrix} w_1 \\ 0 \\ 0 \end{vmatrix}$ $\qquad$ $J_0 \vec{w} = \begin{bmatrix} J_{11} w_1 \\ 0 \\ 0 \end{bmatrix} = J_{11} \begin{bmatrix} w_1 \\ 0 \\ 0 \end{bmatrix}$

<u>ZGLED</u>

$\rightarrow J = \begin{bmatrix} 1 & h & 0 \\ h & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$ → iz tega hočemo dobiti tak tenzor → $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

↓

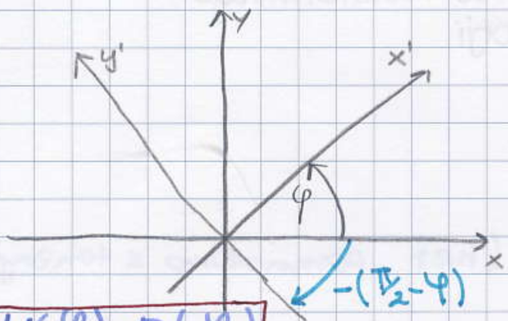
$J_0 \vec{w} = \lambda \vec{w}$

$J_0 = U^{-1} J U$

$\vec{w}_1 = \begin{bmatrix} 0 \\ 0 \\ w_0 \end{bmatrix}$

$J \vec{w}_1 = \begin{bmatrix} 0 \\ 0 \\ k w_0 \end{bmatrix} = k \vec{w}_0$

$R(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$



$\boxed{U(\varphi) = R(-\varphi)}$

$J_0 = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & h & 0 \\ h & 1 & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$J_0 = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c+hs, & -s+hc, & 0 \\ hc+s, & -hs+c, & 0 \\ 0, & 0, & k \end{bmatrix}$

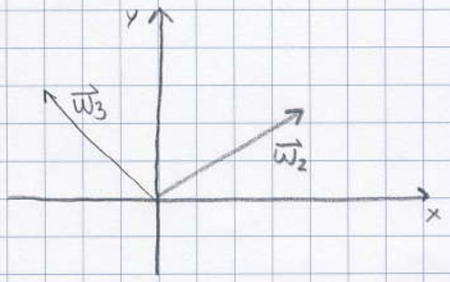
$J_{0\,1,2} = -\cancel{cs} + hc^2 - hs^2 + \cancel{cs}$
$= hc^2 - hs^2$
$= h(c^2 - s^2) = 0$ → to želimo → da dobimo diagonalno matriko z a, b, c
$= h\cos(2\varphi)$ $\qquad \varphi = \frac{\pi}{4}$

$J_{0\,1,1} = c^2 - hcs + hcs + s^2 = \underline{\underline{1+h}}$

$J_{0\,2,2} = s^2 - hcs - hcs + c^2 =$
$= s^2 - 2hcs + c^2 =$
$= 1 - h \cdot \sin(2\varphi) = \underline{\underline{1-h}}$

vrednosti sta neodvisni od h

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 1+h & 0 & 0 \\ 0 & 1-h & 0 \\ 0 & 0 & k \end{bmatrix}$$ → poiskali smo lastni koordinatni sistem



$$\vec{\omega_2} = \begin{Vmatrix} \omega \\ \omega \\ 0 \end{Vmatrix}$$

$$= \omega_2 \begin{Vmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{Vmatrix}$$ → normiran

$$J\vec{\omega_2} = \begin{bmatrix} 1 & h & 0 \\ h & 1 & 0 \\ 0 & 0 & k \end{bmatrix} \cdot \begin{bmatrix} \omega \\ \omega \\ 0 \end{bmatrix} = \begin{bmatrix} \omega(1+h) \\ \omega(h+1) \\ 0 \end{bmatrix} = (1+h) \begin{bmatrix} \omega \\ \omega \\ 0 \end{bmatrix}$$

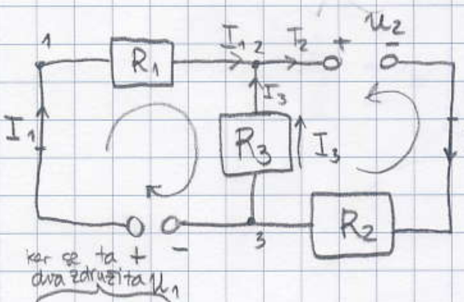
$$\vec{\omega_3} = \begin{Vmatrix} -\omega \\ \omega \\ 0 \end{Vmatrix}$$

→ lastne vrednosti

$$J\vec{\omega_3} = \begin{bmatrix} -\omega + h\omega \\ -h\omega + \omega \\ 0 \end{bmatrix} = (1-h) \begin{bmatrix} -\omega \\ \omega \\ 0 \end{bmatrix}$$

$$J_c = \frac{m}{12}(a^2+b^2)$$
$$J_b = \frac{m}{12}(a^2+c^2)$$
$$J_a = \frac{m}{12}(b^2+c^2)$$

MATRIČNO REŠEVANJE ENAČB



Iščemo kolikšen tok gre skozi vsak upornik.

$I_1 + I_3 = I_2$ (tok k gre vn gre tud not)

$U = -IR$ ..... če gledamo smer toka

$-I_1 R_1 + I_3 R_3 + U_1 = \emptyset$ → ker pridemo v isto točko → za 1 vir

$I_2 R_2 - U_2 + I_3 R_3 = \emptyset$ → za 2 vir

Imamo 3 enačbe in 3 neznanke.

$$I_1 + I_3 = I_2$$
$$-I_1 R_1 + I_3 R_3 + U_1 = 0$$
$$I_2 R_2 - U_2 + I_3 R_3 = 0$$

$$A \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$1 I_1 - 1 \cdot I_2 + 1 I_3 = 0$$

$$-R_1 I_1 + 0 \cdot I_2 + R_3 I_3 = -U_1$$

$$0 I_1 + R_2 I_2 + R_3 I_3 = U_2$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -R_1 & 0 & R_3 \\ 0 & R_2 & R_3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ -U_1 \\ U_2 \end{bmatrix}$$

$$A\vec{\imath} = \vec{b} \qquad \vec{\imath} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$A\vec{\imath} = \vec{b} \;\Big|\; A^{-1} \cdot$$

$$A^{-1} A \vec{\imath} = A^{-1} \vec{b}$$

$$\vec{\imath} = A^{-1} \vec{b}$$

$A^{-1} \rightarrow$ poiskali bomo z determinanto

$$\det A = a_{11} \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{a_{22} \cdot a_{33} - a_{23} a_{32}} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = \begin{cases} = 0 \Rightarrow A^{-1} \text{ ne obstaja} \\ \neq 0 \end{cases}$$

$$A^{-1} = \frac{1}{\det A} \Big| \det A_{11} \; -\det A \; \det A_{13}$$

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{matrix} i \\ j \end{matrix}$$

$\rightarrow$ to pomeni, da izbrišemo i-ti stolpec in j-to vrstico

$$\det A = a_{11} \det A_{11} - a_{1,2} \det A_{21} \times a_{13} \det A_{13}$$

$$C = A^{-1} A = I = \delta_{ij}$$

$$C_{11} = \frac{1}{\det A} \Big[\underbrace{a_{11} \det A_{11} - \det A \; a_{21} + \det A \; a_{31}}_{\det A} \Big] = I$$

$(AB)^T = B^T \cdot A^T$

ROTACIJA ZA φ $\qquad R^{-1}(\varphi) = R(-\varphi) = R^T(\varphi)$

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

MINORJI - če v neki matriki izbrišemo eno vrstico in stolpec in dobimo pomanjšano matriko

$$\begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} = D_{12}$$

pomeni, da smo črtali 1 vrstico in 2 stolpec

$$\boxed{A_{ij} = (-1)^{i+j} \det(D_{ij})}$$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & \ddots & \\ A_{1n} & & \cdots & A_{nn} \end{bmatrix}$$

PRAVILA

① $\det(A \cdot B) = \det A \cdot \det B$

② $\vec{n} \times \vec{n}$ matrika ... $\det A = 0$

$\neq 0 \rightarrow$ stolpci in vrstice morajo biti linearno neodvisni med sabo

$$\underset{\substack{\downarrow \\ \text{tenzor}}}{\underline{A}} \cdot \vec{x} = \lambda \vec{x} = \lambda I \vec{x}$$

$$\lambda I = \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{bmatrix}$$

$(A - \lambda I)\vec{x} = 0$ $\qquad$ Homogen sistem enačb

$\underline{A} - \underline{B} = C \Rightarrow C_{ij} = A_{ij} - B_{ij}$ (vse gre po elementih)

$(A - \lambda I)\vec{x} = 0 \Rightarrow \underline{B}\vec{x} = \emptyset$
- $\det B = 0$
- $\det B \neq 0 \Rightarrow$ obstaja B^{-1}

$\underline{B}\,\vec{x} = \emptyset$

$\det B \neq 0$

$\Downarrow$

obstaja B^{-1}

$\vec{x} = \underline{B}^{-1} \cdot \vec{\emptyset} = 0$

trivialna rešitev

$\det B = 0$

$\Downarrow$

pogoj da obstajajo netrivialne rešitve

To rešujemo, da dobimo na začetku lastne vrednosti, nato pa za lastne vrednosti izračunamo še lastne vektorje.

$$\lambda_i, \vec{x}_i \Rightarrow A\vec{x}_i = \lambda_i \vec{x}_i$$

$$S' : \vec{v} \rightarrow U\vec{v} = \vec{v}'$$

$$\vec{A} \rightarrow UAU^{-1}$$

Tako se transponirajo matrike

$$S : \vec{u} = A \cdot \vec{v}$$

$$S' : U\vec{u} = U(A\vec{v}) = \underbrace{UAU^{-1}}_{A'}\underbrace{U\vec{v}}_{\vec{v}'}$$

$$\vec{u}' = A'\vec{v}'$$

$$A \xrightarrow{S \to S'} A' = \begin{bmatrix} a_{11} & & & & 0 \\ & a_{22} & & & \\ & & a_{33} & & \\ & 0 & & \ddots & \\ & & & & a_{nn} \end{bmatrix}$$

$$\det A' = \det(UAU^{-1}) = \det U \cdot \det A \cdot \det U^{-1} =$$

$$= \det U \cdot \det U^{-1} \cdot \det A =$$

$$= \det(U \cdot U^{-1}) \cdot \det A =$$

$$= \det A$$

$$\det A' = \det A$$

ISKANJE LASTNIH VREDNOSTI

$$\underline{A}\,\vec{x} - \lambda \vec{x} = 0$$

$$\underline{B} = \underline{A} - \lambda\,\underline{Id}$$

$$\underline{A}'\vec{x} - \lambda \vec{x} = 0$$

↳ diagonalna matrika

$$S, S' \Rightarrow \det B = \det B'$$

$$B' = \begin{bmatrix} a_{11}-\lambda & & & 0 \\ & a_{22}-\lambda & & \\ & & \ddots & \\ 0 & & & a_{nn}-\lambda \end{bmatrix}$$

$$\det B' = (a_{11}'-\lambda)(a_{22}'-\lambda)\dots(a_{nn}'-\lambda)$$

$\mapsto$ to pomeni transponiramo

$\lambda_1 = a_{11}$
$\lambda_2 = a_{22}$
$\vec{x}_j = \hat{e}_j$

$$A = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} J = \begin{bmatrix} 0 \\ 0 \\ a_{ij} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} J_{11} & & 0 \\ & J_{22} & \\ 0 & & J_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ 0 \\ 0 \end{bmatrix} = J_{11}\omega_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = J_{11}\vec{\omega}$$

A, A^T

Tenzor vs. matrika
$S \to S'$
A se mora transformirat v $A \to UAU^{-1} \Rightarrow$ če to velja je to tenzor
└ stevke v kvadratni formi

$$A = \underbrace{\tfrac{1}{2}(A+A^T)} + \tfrac{1}{2}(A-A^T)$$

$$b_{ij} = \tfrac{1}{2}(a_{ij} + a_{ji}) = b_{ji}$$

$$\tfrac{1}{2}(A+A^T) = \begin{bmatrix} x & & x \\ & \ddots & \\ & & \end{bmatrix}$$ ... simetrični

$$\underbrace{\tfrac{1}{2}(A-A^T)}$$

$$c_{ij} = \tfrac{1}{2}(a_{ij} - a_{ji}) =$$
$$= \tfrac{1}{2}(a_{ji} - a_{ij}) = -c_{ji}$$

$$\tfrac{1}{2}(A-A^T) = \begin{bmatrix} 0 & 0 & -y \\ y & 0 & \\ & & \ddots \; 0 \end{bmatrix}$$ ... antisimetrični

$$A = \tfrac{1}{2}(A+A^T) + \tfrac{1}{2}(A-A^T) = A_S + A_A$$

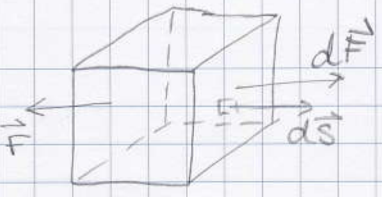
$\hookrightarrow a_{Aij} = -a_{Aji}$

$a_{Aii} = -a_{Aii}$

$$\begin{bmatrix} a & & 0 \\ & a & \\ 0 & & a \end{bmatrix} = a \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} = a\,\mathrm{Id}$$ ... IZOTROPNI $\to$ po diagonali ima vse enake

$$d\vec{F} = -p\,d\vec{S}$$
$$= -p\,\mathrm{I}\,d\vec{S}$$

$$p = \begin{bmatrix} -p & & \\ & -p & \\ & & -p \ddots \end{bmatrix}$$



$$\frac{dl}{l} = \frac{1}{E} \cdot \frac{dF}{S}$$

$$\frac{\Delta x}{h} = \frac{1}{G} \frac{F}{S}$$

$$\vec{F} = \underline{\sigma}\,\vec{S} \qquad \underline{\sigma} = \begin{bmatrix} -p & & \\ & -p & \\ & & -p \end{bmatrix}$$

$$A_S = \frac{1}{3} \ldots$$

$$\underset{\text{sled}}{\mathrm{tr}} A = a_{11} + a_{22} + \ldots + a_{nn} \quad \text{(diagonalni elementi sešteti)}$$

$$A' = UAU^{-1} \Rightarrow \mathrm{tr}A' = \mathrm{tr}A$$

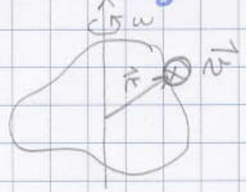
$$A_S = \underbrace{\tfrac{1}{3}\,\mathrm{tr}(A)\,\mathrm{Id}}_{\text{izotropni}} + \underbrace{A - \tfrac{1}{3}\,\mathrm{tr}(A)\,\mathrm{Id}}_{A_{SB}\text{ – simetrično brezsledni}}$$

$$\mathrm{tr}(A_{SB}) = \emptyset$$

Antisimetrični tenzor

Omejimo se na 3 dimenzije.

$$\vec{v} = \vec{\omega} \times \vec{r}$$



$$\downarrow$$

$$\vec{v} = \underline{\Omega} \cdot \vec{r}$$

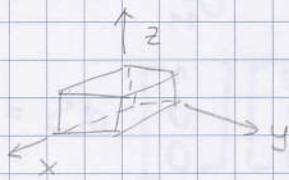
↳ pogledati moramo se vrednosti, elemente tenzorja

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \qquad \begin{aligned} v_1 &= \omega_2 r_3 - \omega_3 r_2 \\ v_2 &= \omega_3 r_1 - \omega_1 r_3 \\ v_3 &= \omega_1 r_2 - \omega_2 r_1 \end{aligned}$$

$$\underbrace{\overset{\underline{\Omega}}{\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}}}_{\text{antisimetričen}} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \omega_2 r_3 - \omega_3 r_2 \\ \omega_3 r_1 - \omega_1 r_3 \\ \omega_1 r_2 - \omega_2 r_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Poglejmo si na primeru delitev na simetričnih tenzorjev.

$$\underline{\sigma} = \begin{bmatrix}\sigma_1 & & 0\\ & \sigma_2 & \\ 0 & & \sigma_3\end{bmatrix} \quad \text{(napetostni tenzor)}$$



$d\vec{F}, d\vec{S}$

$\sigma_{izotropni}$

$$\frac{1}{3}\,tr(\underline{\sigma}) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

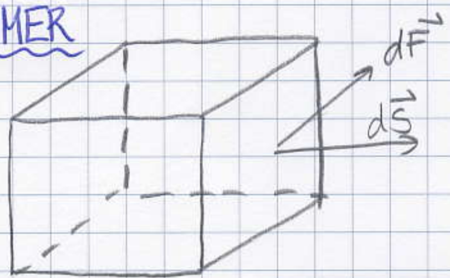
$$\underline{\sigma} = \begin{bmatrix}\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) & & \\ & \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) & \\ & & \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)\end{bmatrix} +$$

$$+ \begin{bmatrix}\sigma_1 - \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) & & \\ & \sigma_2 - \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) & \\ & & \sigma_3 - \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)\end{bmatrix}$$

$$\begin{bmatrix}1 & & \\ & 2 & \\ & & 3\end{bmatrix} = \begin{bmatrix}2 & & \\ & 2 & \\ & & 2\end{bmatrix} + \begin{bmatrix}-1 & & \\ & 0 & \\ & & 1\end{bmatrix}$$

PRIMER

20.4.10



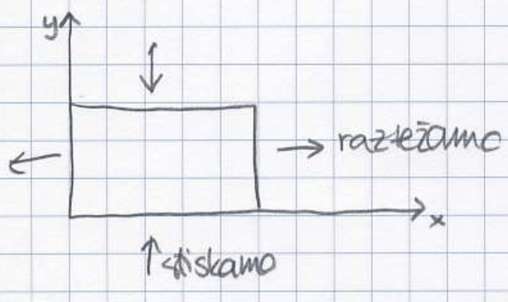
$\sigma_{ij}\, dS_j = dF_i$

napetostni tenzor

$d\vec{F} = \underline{\sigma}\, d\vec{S}$

$d\vec{F} = p\, d\vec{S}$

IZOTROPNI: $p \quad -p\,\underline{Id}$

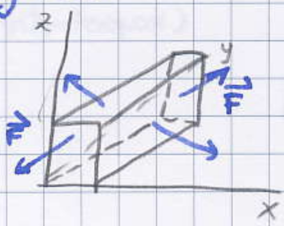


$$\begin{bmatrix}\sigma_{11} & & 0\\ & \sigma_{22} & \\ 0 & & 0\end{bmatrix}$$

Tlak je negativen če vlečemo narazen.

$$\frac{dl}{l} = \frac{1}{E}\,\underbrace{\frac{dF}{S}}_{\sigma_{11}}$$

$$\begin{bmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{ker je normala v smeri } x} dS = \overbrace{\begin{bmatrix} 0 \\ \sigma_{21} \\ 0 \end{bmatrix}}^{\text{sila je v y smeri}} dS$$



To sta strižni sili

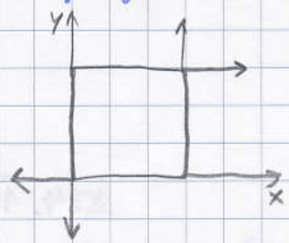
$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sigma_{12} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{12}\, dS \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sigma_{12}\, dS \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Če pogledamo iz vrha

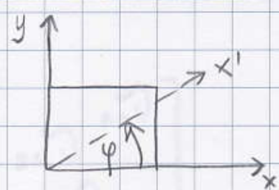


Vsota navorov je 0 → če na enem koncu ne bi bilo puščic, bi se vrtela.

$$\begin{bmatrix} \sigma_{11}^{\circ} & 0 & 0 \\ 0 & \sigma_{22}^{\circ} & 0 \\ 0 & 0 & \sigma_{33}^{\circ} \end{bmatrix}$$

$$A_s = \left(\tfrac{1}{3} s\, tr(A)\, Id\right) + \boxed{A_s - \tfrac{1}{3} s\, |(A)| d}$$

$$\sigma = \begin{bmatrix} \sigma_{11} & & \\ & \sigma_{22} & \\ & & 0 \end{bmatrix}$$



$$\sigma' = U \sigma \underbrace{U^{-1}}_{U^T}$$

$$U = R_z(-\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

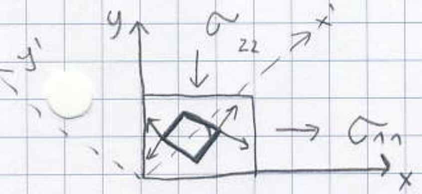
$$U\vec{v} = \vec{v}'$$

$$\sigma \vec{v} = \vec{F}$$
$$\sigma' \vec{v}' = \vec{F}'$$

$$v \to \vec{v}' = U\vec{v}$$
$$\vec{F}' = U\vec{F}$$

$$U\sigma_{\underset{\uparrow}{}} \vec{v} = \vec{F}'$$
$$U^{-1}U$$

$$\underbrace{U\sigma U^{-1}}_{\sigma'} \underbrace{U\vec{v}}_{\vec{v}'} = \vec{F}'$$



$$\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & & \\ & \sigma_{22} & \\ & & 0 \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\sigma_{11} & -s\sigma_{11} & 0 \\ s\sigma_{22} & c\sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} c^2\sigma_{11} + s^2\sigma_{22} & -sc\sigma_{11} + sc\sigma_{22} & 0 \\ -sc\sigma_{11} + sc\sigma_{22} & s^2\sigma_{11} + c^2\sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Recimo, da je $\sigma_{22} = -\sigma_{11}$

$$\sigma' = \begin{bmatrix} \sigma_{11}(c^2 - s^2) & -2sc\sigma_{11} & 0 \\ -2sc\sigma_{11} & \sigma_{11}(s^2 - c^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\varphi = 45°$ $\qquad \cos\varphi = \sin\varphi = \frac{\sqrt{2}}{2}$

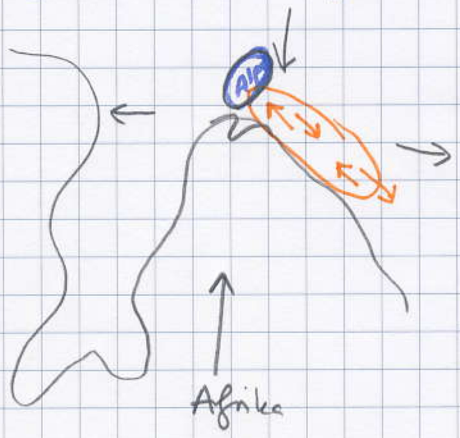
$$\sigma' = \begin{bmatrix} 0 & -\sigma_{11} & 0 \\ -\sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Koordinatni sistem smo zavrteli za $45°$.

$$\sigma' = \begin{bmatrix} 0 & -\sigma_{11} & 0 \\ -\sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ds = \begin{bmatrix} 0 \\ -\sigma_{11}\, ds \\ 0 \end{bmatrix}$$

Zdaj smo izračunali kakšne strižne sile se pojavijo v blokih (beton).

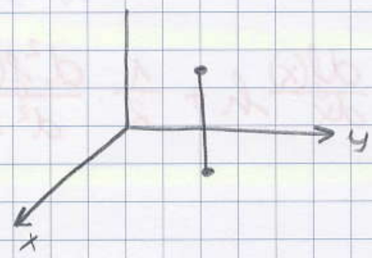
STRIŽNE SILE pri nas



Tak zdaj smo tudi mi podoben izračunali

FUNKCIJE VEČ SPREMENLJIVK

$z = f(x,y) \rightarrow$ 2 spremenljivki



$f(x,y,z) \rightarrow$ funkcija 3 spremenljivk
$\rightarrow$ rabimo 4 dimenzijo

a) PARCIALNI ODVOD

ODVOD

$df = f' dx \quad | \quad f' = \frac{df}{dx}$

$df = df_{y=konstanten} + df_{x=konstanten}$ za funkcije več spremenljivk!

$df = \left(\frac{df}{dx} dx\right)_{y=konst} + \left(\frac{df}{dy} dy\right)_{x=konst}$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$\frac{\partial f}{\partial x}$ ... parcialni odvod

$\frac{df}{dx} = \frac{\partial f}{\partial x} \Big|_{y=konst.}$

$\frac{df}{dy} = \frac{\partial f}{\partial y} \Big|_{x=konst.}$

$f(x,y,z,\ldots)$ $\quad df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

$\hookrightarrow \frac{df}{dx} \Big|_{\substack{y=konst \\ z=konst}}$

$f(x,y) = x^2 y$

$\frac{\partial f}{\partial x} = 2x\,y \qquad \frac{\partial f}{\partial y} = x^2$

b) TAYLORJEVA VRSTA

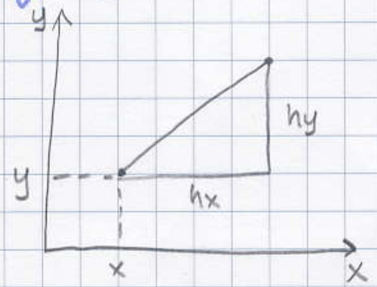
$$f(x+h) = f(x) + \Delta f$$

$$= f(x) + f'(x)\,\Delta x$$

$\Delta x = h$ → če je h majhen

$$\boxed{f(x+h) = f(x) + \frac{df(x)}{dx}h + \frac{1}{2}\frac{d^2f(x)}{d^2x}h^2 + \dots + \frac{1}{n!}\frac{d^nf(x)}{d^nx}h^n}$$

→ $f(x,y)$



$$f(x+h_x, y+h_y) = f(x,y) + \Delta f =$$

$$= f(x,y) + \frac{\partial f}{\partial x}h_x + \frac{\partial f}{\partial y}h_y + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x}h_x^2 + \frac{1}{2}\frac{\partial^2 f}{\partial^2 y}h_y^2 + \frac{\partial^2 f}{\partial x \partial y}h_x h_y$$

PRAVILO!

→ $$\Phi_v = \text{konst.}\,\frac{r^4 \Delta p}{l\,\eta}$$

Naše spremenljivke so: $\Phi_v(r, \Delta p, l, \eta)$

$$d\Phi_v = \underbrace{k\frac{\Delta p}{l\eta}4r^3}_{\text{odvod po } r}\underset{\Delta r}{dr} + \overbrace{k\frac{r^4}{l\eta}}^{\text{odvod po } \Delta p}\underset{\delta\Delta p \,\to\, \text{sprememba spremembe tlakov}}{d\Delta p} + \overbrace{k\frac{r^4\Delta p}{\eta}(-l^{-2})}^{\text{odvod po } l}\underset{\Delta l}{dl} - \overbrace{k\frac{r^4\Delta p}{l}\eta^{-2}}^{\text{odvod po } \eta}\underset{\Delta\eta}{d\eta}$$

BOJANOV TRIK

$$\Phi_v = k\frac{r^4\Delta p}{l\eta} \quad \Big|\cdot \ln$$

$$\ln\Phi_v = \ln r^4 + \ln\Delta p - \ln l - \ln\eta \quad \Big|\cdot d$$

$$\frac{d\phi_v}{\phi_v} = \frac{4dr}{r} + \frac{d\Delta p}{\Delta p} - \frac{dl}{l} - \frac{d\eta}{\eta}$$

$$d \ln f = \frac{1}{f} df$$

$\rightarrow z = x + y$

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2$$

$$z \pm \Delta z = x \pm \Delta x + y \pm \Delta y$$

$f(x,y)$

$$df(x,y) = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy$$

$$\downarrow \sigma_f \qquad \downarrow \sigma_x \qquad \downarrow \sigma_y$$

$$\boxed{\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)\sigma_y^2}$$

$\rightarrow f(x,y) = x^2 + y$

$$\sigma_f^2 = (2x)^2 \sigma_x^2 + \sigma_y^2$$
$$= 4x^2 \sigma_x^2 + \sigma_y^2$$

$\rightarrow a = \frac{2h}{t^2}$

$$da = \frac{1}{t^2} 2\,dh - 2h \cdot 2\, t^{-3}\, dt$$

$$\sigma_a^2 = \left(\frac{2}{t^2}\right)^2 \sigma_h^2 + \left(\frac{4h}{t^3}\right)^2 \sigma_t^2$$

$\rightarrow$ z LOGARITMI

$$\ln a = \ln 2 + \ln h - 2 \ln t \,|\, d$$

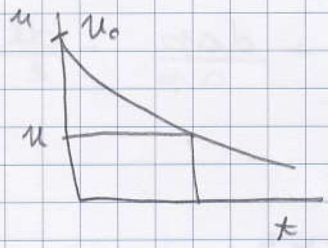
$$\frac{da}{a} = \frac{dh}{h} - 2\frac{dt}{t}$$

$$\downarrow$$

$$\left(\frac{\sigma_a^2}{a^2}\right) = \left(\frac{\sigma_h}{h}\right)^2 + 4\left(\frac{\sigma_t}{t}\right)^2$$

$\rightarrow u = u_0 e^{-\frac{t}{\tau}}$

Spremenljivke: u, u_0, t
$\sigma_u, \sigma_{u_0}, \sigma_t$



① $\ln u = \ln u_0 - \frac{t}{\tau} \ln e$

$$\frac{t}{\tau} = \ln\left(\frac{u_0}{u}\right)$$

$$\tau = \frac{t}{\ln\left(\frac{u_0}{u}\right)}$$

② $\ln u = \ln u_0 - \left(\frac{t}{\tau}\right) \Big| \cdot d$ Totalni diferencial

obe količini sta spremenljivki

$$\frac{du}{u} = \frac{du_0}{u_0} - \frac{1}{\tau} dt + t\,\tau^{-2} d\tau \qquad d_x \rightarrow \sigma_x$$

$$\left(\frac{\sigma_u}{u}\right)^2 = \left(\frac{\sigma_{u_0}}{u_0}\right)^2 + \frac{1}{\tau^2}\sigma_t^2 + \left(\frac{t}{\tau^2}\right)^2 \sigma_\tau^2$$

→ Tako bi bilo, če bi hotel računati $u \rightarrow$ njegovo napako

τ moramo nesti na drugo stran.

$$\frac{t}{\tau^2} d\tau = \frac{du}{u} - \frac{du_0}{u_0} + \frac{1}{\tau} dt$$

$$\left(\frac{t}{\tau^2}\right)^2 d\tau = \left(\frac{\sigma_u}{u}\right)^2 + \left(\frac{\sigma_{u_0}}{u_0}\right)^2 + \frac{1}{\tau^2}\sigma_t^2$$

$\rightarrow V = V(T, p)$

$$dV = \underbrace{\frac{\partial V}{\partial T}\Big|_{p=\text{konst}} dT}_{\beta V dT} + \underbrace{\frac{\partial V}{\partial p} dp}_{-\chi V dp} \Big| : V$$

$\frac{\partial V}{\partial T}$

$\Delta V = \beta V \Delta T$ p = konst.

$\Delta V = -\chi V \Delta p$ T = konst.
└ stisljivost

$$\frac{dV}{V} = \beta dT - \chi dp$$

c) Ekstremi

$f(x)$ ima ekstrem, če velja $\frac{df}{dx} = 0$

$\frac{d^2 f}{dx} > 0$ minimum

< 0 maksimum

$= 0$ sedlo

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Ekstremi za funkcije več spremenljivk: $\frac{\partial f}{\partial x} = 0 \overset{\text{hkrati}}{\wedge} \frac{\partial f}{\partial y} = 0$

$$\frac{\partial'' f}{\partial'' x} > 0 \wedge \frac{\partial'' f}{\partial'' y} > 0 \Rightarrow \min$$

Vezani ekstrem

$f(x,y) = \text{ekstrem}$

$x^2+y^2=r^2 \longrightarrow G(x,y) = x^2+y^2-r^2$

$f(x,y) = \text{ekstrem}$

$G(x,y) = 0$ vez – dodaten pogoj, kje iskati ekstrem

REŠEVANJE:

① Lahko iz vezi izpostavimo eno spremenljivko:

$$y = \sqrt{x^2+r^2} \rightarrow f(x, \sqrt{r^2-x^2})$$

Vendar to vedno ne moremo storiti, saj so kdaj vezi zelo zakomplicirane. (npr $y \operatorname{tg} y = x \cos x$).

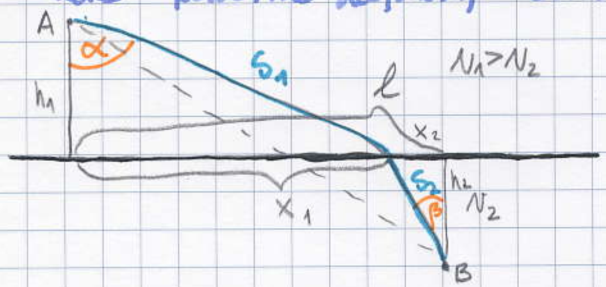
② LAGRANGEOVI MULTIPLIKATORJI

$$\tilde{f}(x,y,\lambda) = f(x,y) + \lambda \overset{\text{vez}}{G(x,y)}$$

zahtevamo: $\frac{\partial \tilde{f}}{\partial x} = 0 \wedge \frac{\partial \tilde{f}}{\partial y} \wedge \frac{\partial \tilde{f}}{\partial \lambda} = 0 = G(x,y)$

PRIMER:

Če hočemo priděti nekam Bogu za ledžama, gremo lahko po avtocesti in je daljša pot, ali po krajši poti po cestah. Kako porabimo najmanj časa?



Ta je časovno najkrajša.

$$t = \frac{s_1}{v_1} + \frac{s_2}{v_2} = \min \quad *$$

$x_1 + x_2 = l$ VEZ

① $\sqrt{s_1^2 - h_1^2} + \sqrt{s_2^2 + h_2^2} = \ell$

② $t = \frac{\sqrt{x_1^2 + h_1^2}}{v_1} + \dots$

Vse funkcije, ki imajo korene je komplicirano odvajati.

③ $s_1 = \frac{h_1}{\cos\alpha}$

$s_2 = \frac{h_2}{\cos\beta}$

$x_1 = h_1 \operatorname{tg}\alpha$

$x_2 = h_2 \operatorname{tg}\beta$

$$\tilde{f} = \underbrace{\frac{h_1}{v_1\cos\alpha} + \frac{h_2}{v_2\cos\beta}}_{*} + \lambda\left(h_1\operatorname{tg}\alpha + h_2\operatorname{tg}\beta - \ell\right)$$

$$\frac{\partial\tilde{f}}{\partial\alpha} = +\frac{\cancel{h_1}\sin\alpha}{v_1\cos^{\cancel{2}}\alpha} + \lambda\cancel{h_1}\frac{1}{\cancel{\cos^2\alpha}} = 0 \Rightarrow \frac{\sin\alpha}{v_1} + \lambda = 0 \Rightarrow \lambda = -\frac{\sin\alpha}{v_1}$$

$$\frac{\partial\tilde{f}}{\partial\beta} = \frac{\cancel{h_2}\sin\beta}{v_2\cos^{\cancel{2}}\beta} + \lambda\cancel{h_2}\frac{1}{\cancel{\cos^2\beta}} = 0 \Rightarrow \frac{\sin\beta}{v_2} + \lambda = 0 \Rightarrow \lambda = -\frac{\sin\beta}{v_2}$$

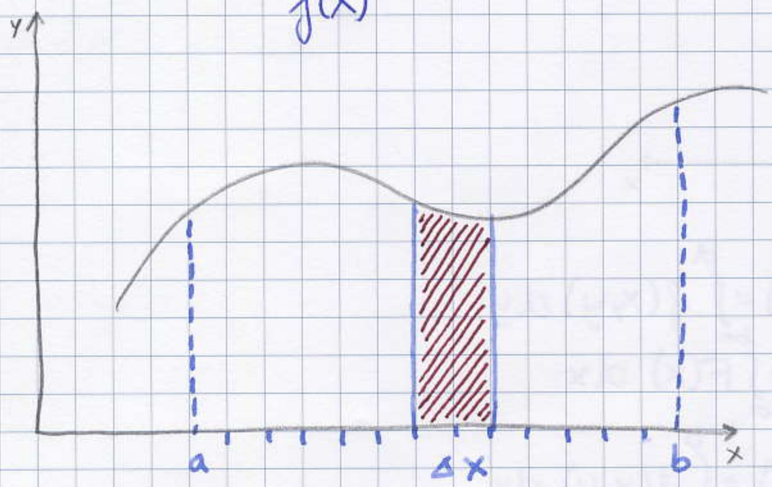
izenačimo $*$

$h_1\operatorname{tg}\alpha + h_2\operatorname{tg}\beta - \ell = 0$

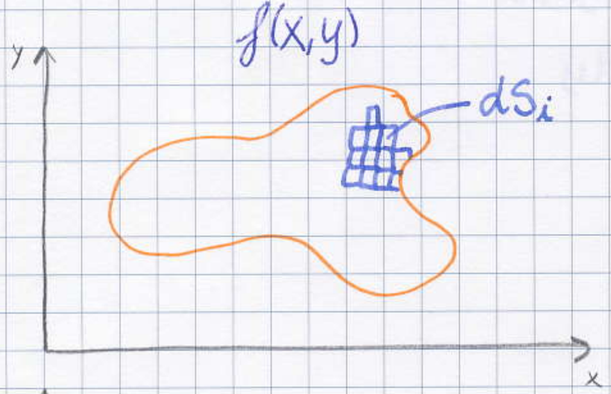
$*$ $\boxed{\frac{\sin\alpha}{v_1} = \frac{\sin\beta}{v_2}}$ Lomni zakon

DVOJNI in MNOGOTERNI INTEGRAL

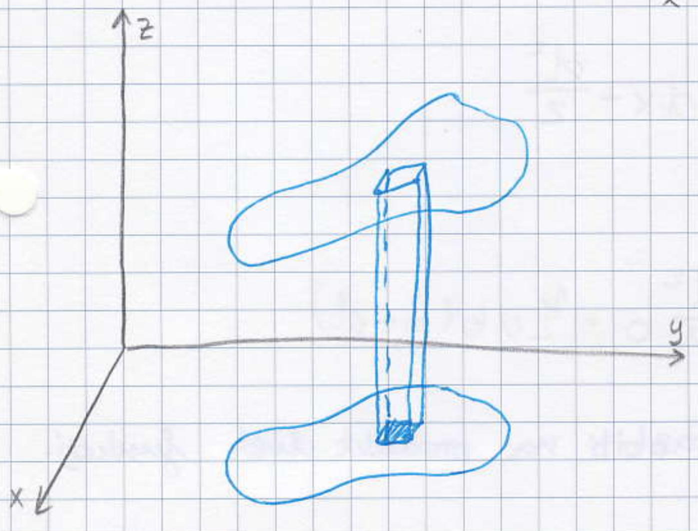
11.5.10



$$\lim_{\Delta x\to 0}\sum_i f(x_i)\,\Delta x = \int_a^b f(x)\,dx$$



$$\lim_{\Delta S_i\to 0}\sum f(x_i,y_i)\,\Delta S_i = \iint f(x,y)\,dS$$



Lepe lastnosti integriranja:

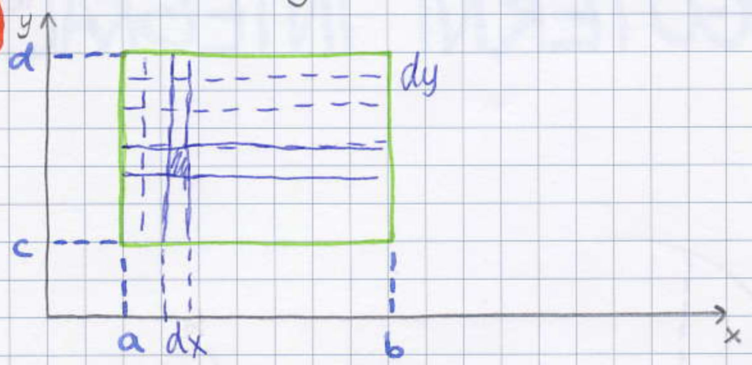
① $\iint [f(x,y) + g(x,y)]\,dS = \iint f(x,y)\,dS + \iint g(x,y)\,dS$

② $\iint A f(x,y)\,dS = A\iint f(x,y)\,dS$

③ $\iint_D f(x,y)\,dS = \iint_{D_1} f(x,y)\,dS + \iint_{D_2} f(x,y)\,dS$

Računanje

1



$$I = \int_a^b \int_c^d f(x,y)\,dx\,dy = \begin{cases} \text{a)}\ F(x) = \int_c^d f(x,y)\,dy \\ \quad I = \int_a F(x)\,dx \\ \text{b)}\ F(x) = \int_a^b f(x,y)\,dx \\ \quad I = \int_c^d F(x)\,dy \end{cases}$$

ZGLED

① $a = c = 0$

$$\int_0^b \int_0^d (x+y)\,dx\,dy$$

$$F(x) = \int_0^d (x+y)\,dy = xy + \frac{y^2}{2}\Big|_0^d = d\cdot x + \frac{d^2}{2}$$

$$I = \int_0^b \left(dx + \frac{d^2}{2}\right) dx =$$

$$= d\frac{x^2}{2} + \frac{d^2}{2}x\Big|_0^b = d\frac{b^2}{2} + \frac{d^2}{2}b = \tfrac{1}{2}db(b+d)$$

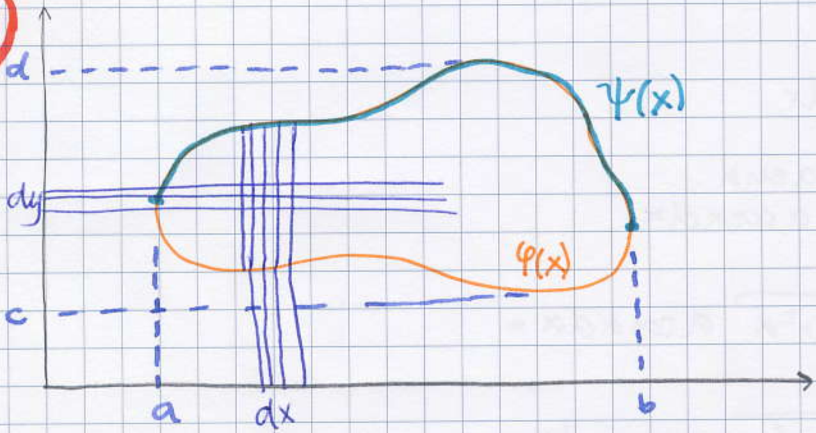
② $f(x,y)$ je separabilna – da se jo razbiti na produkt dveh funkcij.

$$f(x,y) = g(x)\,h(y)$$

$$\iint f(x,y)\,dx\,dy = \iint g(x)\,h(x)\,dx\,dy$$

$$= \int dx\, g(x) \int dy\, h(y)$$

2



Lik lahko razdelimo z dvema funkcijama.

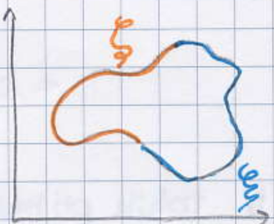
$$I = \underbrace{\int_a^b dx \int_{\varphi(x)}^{\psi(x)} dy\, f(x,y)}_{\text{če integriramo na začetku po } x} = \underbrace{\int_c^a dy \int_{\zeta(y)}^{\xi(y)} dx\, f(x,y)}_{\text{če gledamo na začetku po } y}$$

TUDI MEJE SO ODVISNE OD X-a!!!

če integriramo na začetku po X

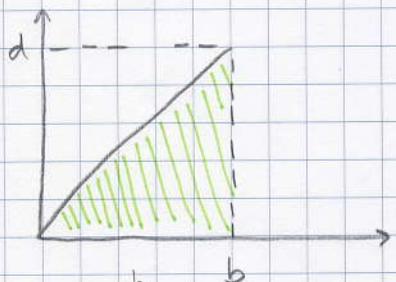
če gledamo na začetku po y

AMPAK POTEM



ZGLED

①



$\varphi(x) = 0$

$\psi(x) = k\,x, \quad k = \frac{d}{b}$

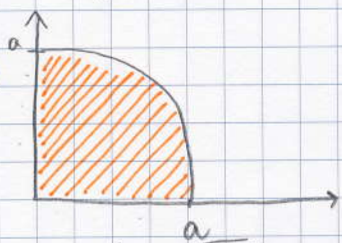
$f(x) = xy^2$

$$I = \int_0^b dx \int_0^{kx} xy^2\, dy$$

$$= \int_0^b dx \left[x \frac{y^3}{3}\right]\Big|_0^{kx}$$

$$= \int_0^b dx\, x \frac{(kx)^3}{3} = \frac{k^3}{3}\int_0^b dx\, x^4$$

②



$\varphi(x) = 0$

$\psi(x) = \sqrt{a^2 - x^2}$

$f(x,y) = k$

$$I = \int_0^a dx \int_0^{\sqrt{a^2-x^2}} k\, dy =$$

$$= \int_0^a dx\, k \cdot y \Big|_0^{\sqrt{a^2-x^2}} = \int_0^a dx\, k \sqrt{a^2-x^2} =$$

$$= k \int_0^a \sqrt{a^2-x^2}\, dx$$

$x = a \sin\alpha$
$dx = a \cos\alpha\, d\alpha$

$$= k \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \sin^2\alpha}\; a\cos\alpha\, d\alpha =$$

$$= k \int_0^{\frac{\pi}{2}} a\sqrt{1-\sin^2\alpha}\; a\cos\alpha\, d\alpha =$$

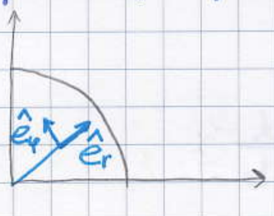
$$= k \int_0^{\frac{\pi}{2}} a^2 \cos^2\alpha\, d\alpha =$$

$$= k a^2 \int^{\frac{\pi}{2}} \cos^2\alpha\, d\alpha = \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

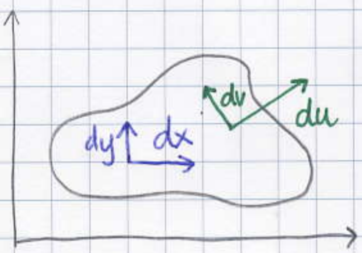
$$= k a^2 \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\alpha}{2}\, d\alpha =$$

$$= k \frac{a^2}{2} \alpha + 0 = k \frac{a^2}{4} \pi$$

V takih primerih potem raje delamo z polarnimi koordinatami.



(3)



$r, \varphi \leftarrow x, y$

$x = r\cos\varphi$
$y = r\sin\varphi$

$r = \sqrt{x^2+y^2}$

$\operatorname{tg}\varphi = \frac{y}{x}$

$u(x,y)$
$v(x,y)$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

enotski vektor v smeri x

$$dx\,\hat{e}_x \times dy\,\hat{e}_y$$

$$\begin{bmatrix} dx \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ dy \\ 0 \end{bmatrix}$$

$$du\,\hat{e}_u = \frac{\partial u}{\partial x} dx\,\hat{e}_x + \frac{\partial u}{\partial y} dy\,\hat{e}_y = \begin{bmatrix} \frac{\partial u}{\partial x} dx \\ \frac{\partial u}{\partial y} dy \\ 0 \end{bmatrix}$$

$$d\vec{S} = \begin{bmatrix} \frac{\partial u}{\partial x} dx \\ \frac{\partial u}{\partial y} dy \\ 0 \end{bmatrix} \times \begin{bmatrix} \frac{\partial v}{\partial x} dx \\ \frac{\partial v}{\partial y} dy \\ 0 \end{bmatrix} =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial u}{\partial x} dx & \frac{\partial u}{\partial y} dy & 0 \\ \frac{\partial v}{\partial x} dx & \frac{\partial v}{\partial y} dy & 0 \end{vmatrix} = \left(0, 0, \frac{\partial u}{\partial x} dx \frac{\partial v}{\partial y} dy - \frac{\partial v}{\partial x} dx \frac{\partial u}{\partial y} dy\right)$$

$$= \vec{k} \det \begin{bmatrix} \frac{\partial u}{\partial x} dx & \frac{\partial u}{\partial y} dy \\ \frac{\partial v}{\partial x} dx & \frac{\partial v}{\partial y} dy \end{bmatrix} =$$

$$= \vec{k} \underbrace{\det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}}_{\text{Jacobijeva determinanta}} \underbrace{dx\,dy}_{dS'}$$

Vedno vzamemo za ploščino absolutno vrednost.

$$du\,dv = \left| \quad \right| dx\,dy$$

$$dx = \cos\varphi\, dr + r(-\sin\varphi)\, d\varphi$$
$$dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi$$

$$\begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

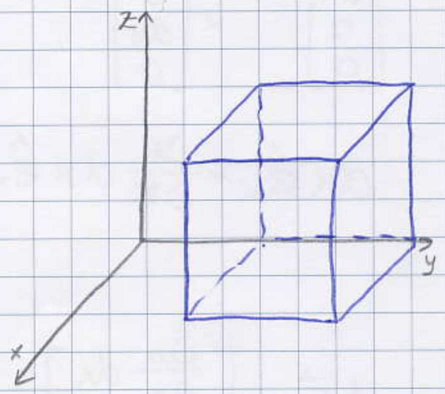
$$dx\,dy = r \cdot dr\, d\varphi$$

$$\iint_D k\,dx\,dy = \int_0^a\int_0^{\frac{\pi}{2}} k\, r\,dr\,d\varphi$$
$$= k\int_0^a r\,dr\int_0^{\frac{\pi}{2}} d\varphi = k\,\frac{a^2}{2}\cdot\frac{\pi}{2}$$

③ $$I = \iiint f(x,y,z)\,dV = \iiint f(x,y,z)\,dx\,dy\,dz =$$
$$= \int_a^b dx \int_{\alpha(x)}^{\beta(x)} dy \underbrace{\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz}_{F(x,y)}$$

$$\underbrace{\int_{\alpha(x)} F(x,y)}_{\int_{\alpha(x)}^{\beta(x)} dy\,F(x,y)}$$

$$\underbrace{\int_{\alpha(x)}^{\beta(x)} dy\,F(x,y)}_{\int dx\, G(x) = I}$$

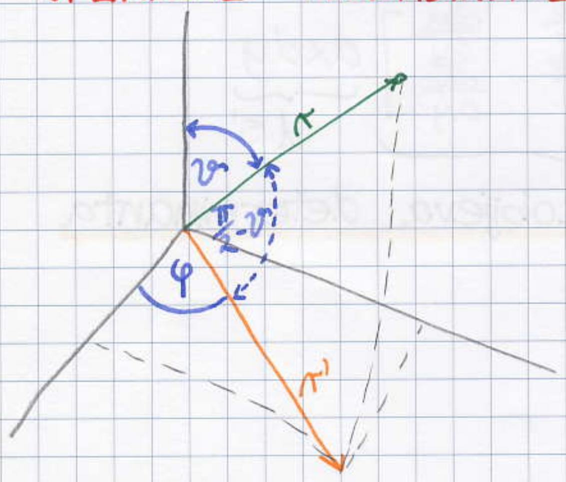


CILINDRIČNE KOORDINATE

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$

$$dV = dx\,dy\,dz = r\,dr\,d\varphi\,dz$$

SFERIČNE KOORDINATE



$x = r'\cos\varphi = r\sin\vartheta\cos\varphi$
$y = r'\sin\varphi = r\sin\vartheta\sin\varphi$
$z = r\cos\vartheta$

$dx = \sin\vartheta\cos\varphi\,dr + r\cos\vartheta\cos\varphi\,d\vartheta + r\sin\vartheta(-\sin\varphi)\,d\varphi$

$dy = \sin\vartheta\sin\varphi\,dr + r\cos\vartheta\sin\varphi\,d\vartheta + r\sin\vartheta\cos\varphi\,d\varphi$

$dz = \cos\vartheta \quad + r\sin\vartheta\,d\vartheta + 0$

JACOBIJEVE DETERMINANTA

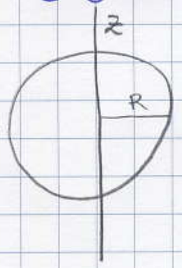
$$\begin{vmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & r\sin\vartheta & 0 \end{vmatrix} = \cos\vartheta\,[\,r\cos\vartheta\overbrace{\sin\vartheta(\cos^2\varphi+\sin^2\varphi)}^{r}\,]$$
$$+\, r\sin\vartheta\,[\,r\sin^2\vartheta\,] =$$

$$\boxed{= r^2\sin\vartheta}$$

JACOBNEVA DETERMINANTA V TREH DIMENZIJAH

$$dx\,dy\,dz = r^2 \sin\vartheta \, dr\, d\vartheta\, d\varphi$$

ZGLED



$$I_z = \iiint (x^2+y^2)\rho \, dV$$

$$I = \rho \iiint x^2 \, dx\,dy\,dz$$

$$= \rho \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin^2\vartheta \cos^2\varphi \, r^2 dr \sin\vartheta \, d\vartheta \, d\varphi$$

$$= \rho \underbrace{\int_0^R r^4 dr}_{\frac{R^5}{5}} \underbrace{\int_0^\pi \sin^3\vartheta \, d\vartheta}_{4/3} \underbrace{\int_0^{2\pi} \cos^2\varphi \, d\varphi}_{\pi}$$

$$\vartheta: \int \sin^2\vartheta \sin\vartheta \, d\vartheta =$$

$$= \int (1-\cos^2\vartheta)\sin\vartheta \, d\vartheta \overset{*}{=}$$

$$u = \cos\vartheta$$
$$du = -\sin\vartheta \, d\vartheta$$

$$\overset{*}{=} -\int_1^{-1} (1-u^2)\, du = \int_{-1}^{1} (1-u^2)\, du =$$

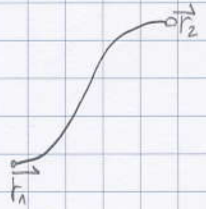
$$= u - \frac{u^3}{3} \Big|_{-1}^{1}$$

$$= \frac{2}{5} MR^2$$

18.5.10

INTEGRAL PO POTI, PO PLOSKVI

$$A = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\, d\vec{r}$$



$$U = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\, d\vec{s}$$

$$\oint \vec{B}\, d\vec{s} = I$$ Amperov zakon

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{d\vec{s} \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

$\vec{r} = (x, y, z)$ $\vec{r}(t)$ - lega točke, odvisna tudi od časa

$\vec{r}(t) = (x(t), y(t), z(t))$

ZGLED

① $t = x$

$x(t) = x$

$y(t) = kx + n$ $z(t) = 0$ $\rightarrow$ premica v ravnini x, y

$y(t) = kx + n$ $z(t) = hx + m$ $\rightarrow$ krivulja v prostoru

$$\left. \begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned} \right|^2$$

$x^2 + y^2 = r^2 \rightarrow$ krožnica če $z(\varphi) = 0$

└ če $z(\varphi) = n \rightarrow$ valj (plašč valja)

$$\int \vec{F}(x,y,z)\, d\vec{s} = \int \left(F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt}\right) dt$$

$$d\vec{s} = \vec{r}(t + dt) - \vec{r}(t) = d\vec{r}$$

$$d\vec{r} = (dx, dy, dz)$$

$$= \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

VIJAČNICA $\qquad t=\varphi$

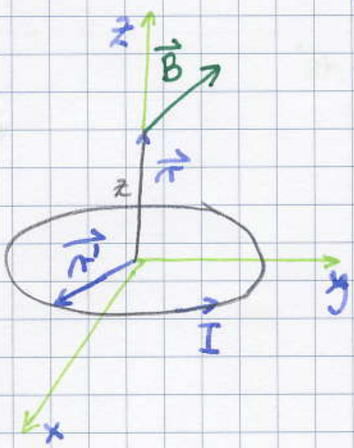
$$d\vec{s} = (-r\sin\varphi,\ r\cos\varphi,\ k)\,d\varphi$$

$$ds = \sqrt{r^2+k^2}\,d\varphi$$

$$s = \int_{\vec{r}_1}^{\vec{r}_2} ds \longrightarrow \text{NI VEKTORJEV}$$

Dolžina vijačnice

VIJAČNICA $\quad s = \int_{\varphi_1}^{\varphi_2} ds = \int_{\varphi_1}^{\varphi_2} \sqrt{r^2+k^2}\,d\varphi = \sqrt{r^2+k^2}\,(\varphi_2-\varphi_1)$



$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{s}\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$\vec{r}\,' = (r_0\cos\varphi,\ r_0\sin\varphi,\ 0) \qquad 0<\varphi<2\pi$$
$$\vec{r} = (0,0,z)$$
$$d\vec{s} \equiv d\vec{r}\,' = (-r_0\sin\varphi,\ r_0\cos\varphi,\ 0)$$
$$\vec{r}-\vec{r}\,' = (-r_0\cos\varphi,\ -r_0\sin\varphi,\ z)$$
$$|\vec{r}-\vec{r}\,'|^3 = \sqrt{r_0^2+z^2}^{\,3}$$

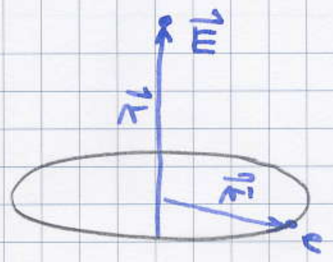
$$d\vec{s}\times(\vec{r}-\vec{r}\,') = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r_0\sin\varphi & r_0\cos\varphi & 0 \\ -r_0\cos\varphi & -r_0\sin\varphi & z \end{vmatrix} d\varphi$$

$$= (r_0\cos\varphi\cdot z,\ r_0\sin\varphi\, z,\ r_0^2\sin^2\varphi + r_0^2\cos^2\varphi)\,d\varphi$$
$$= (r_0\cos\varphi\, z,\ r_0\sin\varphi\, z,\ r_0^2)\,d\varphi$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{1}{\sqrt{r_0^2+z^2}^{\,3}}\Big[\underbrace{\vec{i}\, r_0 z\cos\varphi}_{0} + \underbrace{\vec{j}\, r_0 z\sin\varphi}_{0} + \vec{k}\, r_0^2\Big]\,d\varphi$$

če integriramo

$$= \frac{\mu_0 I}{4\pi}\,\frac{\vec{k}}{\sqrt{r_0^2+z^2}^{\,3}}\,r_0^2$$



$$\vec{E} = \int \frac{de}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

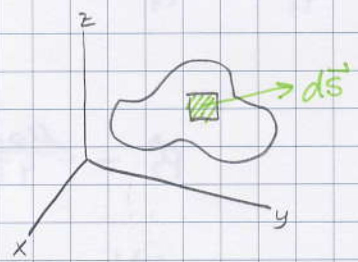
$$de = \rho_e S\, ds$$

$$ds = |d\vec{s}| = r_0\, d\varphi$$

$$\vec{E} = \int \frac{\rho_e S r_0\, d\varphi}{\sqrt{r_0^2+z^2}^3}(-r_0\cos\varphi, -r_0\sin\varphi, z) =$$

$$= \frac{\rho_e S r_0\, d\varphi\, z}{\sqrt{r_0^2+z^2}^3}\, 2\pi = \frac{e z}{\sqrt{r_0^2+z^2}^3}$$

Integral po ploskvi



$$\Phi_v = \iint \vec{v}\, d\vec{S}$$

$$d\vec{S} = (dS_x, dS_y, dS_z)$$

$\vec{S}$ leži v ravnini $xy \rightarrow d\vec{S} = (0, 0, dx\,dy)$

$x,z \rightarrow d\vec{S} = (0, dx\,dz, 0)$

$y,z \rightarrow d\vec{S} = (dy\,dz, 0, 0)$

$\alpha = \text{kot}(d\vec{S}, \vec{x})$

$d\vec{S} = (\cos\alpha\, dy\,dz, \cos\beta\, dx\,dz, \cos\gamma\, dx\,dy)$

$d\vec{S} = \vec{n}\, dS \qquad |\vec{n}| = 1$

$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma) = (n_x, n_y, n_z)$