30th International Physics Olympiad

Padua, Italy

Theoretical competition

Thursday, July 22nd, 1999

Please read this first:

1. The time available is 5 hours for 3 problems.
2. Use only the pen provided.
3. Use only the **front side** of the provided sheets.
4. In addition to the problem texts, that contain the specific data for each problem, a sheet is provided containing a number of general physical constants that may be useful for the problem solutions.
5. Each problem should be answered on separate sheets.
6. In addition to "blank" sheets where you may write freely, for each problem there is an *Answer sheet* where you **must** summarize the results you have obtained. Numerical results must be written with as many digits as appropriate to the given data; don't forget the units.
7. Please write on the "blank" sheets whatever you deem important for the solution of the problem, that you wish to be evaluated during the marking process. However, you should use mainly equations, numbers, symbols, figures, and use *as little text as possible*.
8. **It's absolutely imperative** that you write on top of *each* sheet that you'll use: your name ("**NAME**"), your country ("**TEAM**"), your student code (as shown on the identification tag, "**CODE**"), and additionally on the "blank" sheets: the problem number ("**Problem**"), the progressive number of each sheet (from 1 to *N*, "**Page n.**") and the total number (*N*) of "blank" sheets that you use and wish to be evaluated for that problem ("**Page total**"). It is also useful to write the section you are answering at the beginning of each such section. If you use some sheets for notes that you don't wish to be evaluated by the marking team, just put a large cross through the whole sheet, and don't number it.
9. When you've finished, turn in all sheets <u>in proper order</u> (for each problem: answer sheet first, then used sheets in order; unused sheets and problem text at the bottom) and put them all inside the envelope where you found them; then leave everything on your desk. You are not allowed to take **any** sheets out of the room.

This set of problems consists of 13 pages (including this one, the answer sheets and the page with the physical constants)

These problems have been prepared by the Scientific Committee of the 30th IPhO, including professors at the Universities of

Problem 1

Absorption of radiation by a gas

A cylindrical vessel, with its axis vertical, contains a molecular gas at thermodynamic equilibrium. The upper base of the cylinder can be displaced freely and is made out of a glass plate; let's assume that there is no gas leakage and that the friction between glass plate and cylinder walls is just sufficient to damp oscillations but doesn't involve any significant loss of energy with respect to the other energies involved. Initially the gas temperature is equal to that of the surrounding environment. The gas can be considered as perfect within a good approximation. Let's assume that the cylinder walls (including the bases) have a very low thermal conductivity and capacity, and therefore the heat transfer between gas and environment is very slow, and can be neglected in the solution of this problem.

Through the glass plate we send into the cylinder the light emitted by a constant power laser; this radiation is easily transmitted by air and glass but is completely absorbed by the gas inside the vessel. By absorbing this radiation the molecules reach excited states, where they quickly emit infrared radiation returning in steps to the molecular ground state; this infrared radiation, however, is further absorbed by other molecules and is reflected by the vessel walls, including the glass plate. The energy absorbed from the laser is therefore transferred in a very short time into thermal movement (molecular chaos) and thereafter stays in the gas for a sufficiently long time.

We observe that the glass plate moves upwards; after a certain irradiation time we switch the laser off and we measure this displacement.

1. Using the data below and - if necessary - those on the sheet with physical constants, compute the temperature and the pressure of the gas after the irradiation. *[2 points]*
2. Compute the mechanical work carried out by the gas as a consequence of the radiation absorption. *[1 point]*
3. Compute the radiant energy absorbed during the irradiation. *[2 points]*
4. Compute the power emitted by the laser that is absorbed by the gas, and the corresponding number of photons (and thus of elementary absorption processes) per unit time. *[1.5 points]*
5. Compute the efficiency of the conversion process of optical energy into a change of mechanical potential energy of the glass plate. *[1 point]*

Thereafter the cylinder axis is slowly rotated by 90°, bringing it into a horizontal direction. The heat exchanges between gas and vessel can still be neglected.

6. State whether the pressure and/or the temperature of the gas change as a consequence of such a rotation, and - if that is the case – what is its/their new value. *[2.5 points]*

Data

Room pressure: p_0 = 101.3 kPa
Room temperature: T_0 = 20.0°C
Inner diameter of the cylinder: 2r = 100 mm
Mass of the glass plate: m = 800 g
Quantity of gas within the vessel: n = 0.100 mol
Molar specific heat at constant volume of the gas: c_V = 20.8 J/(mol K)
Emission wavelength of the laser: λ = 514 nm
Irradiation time: Δt = 10.0 s
Displacement of the movable plate after irradiation: Δs = 30.0 mm

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Problem	1
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Answer sheet

In this problem you are requested to give your results both as analytical expressions and with numerical data and units: write expressions first and then data (e.g. $A=bc=1.23\ m^2$).

1. Gas temperature after the irradiation …………………………………………………………...

Gas pressure after the irradiation ………………………………………………………………….

2. Mechanical work carried out ……………………………………………………………………...

3. Overall optical energy absorbed by the gas …………………………………………………

4. Optical laser power absorbed by the gas …………………………………………………………

Absorption rate of photons (number of absorbed photons per unit time) …………………...

5. Efficiency in the conversion of optical energy into change of mechanical potential energy of the glass plate ………………………………………………………………….

6. Owing to the cylinder rotation, is there a pressure change? YES 1 NO 1

If yes, what is its new value? ……………………………………………………………

Owing to the cylinder rotation, is there a temperature change? YES 1 NO 1

If yes, what is its new value? ……………………………………………………………

Physical constants and general data

In addition to the numerical data given within the text of the individual problems, the knowledge of some general data and physical constants may be useful, and you may find them among the following ones. These are nearly the most accurate data currently available, and they have thus a large number of digits; you are expected, however, to write your results with a number of digits that must be appropriate for each problem.

Speed of light in vacuum: c = 299792458 m s^{-1}

Magnetic permeability of vacuum: $\mu_0 = 4\pi\ 10^{-7}$ H m^{-1}

Dielectric constant of vacuum: ε_0 = 8.8541878 pF m^{-1}

Gravitational constant: $G = 6.67259\ 10^{-11}$ m^3/(kg s_)

Gas constant: R = 8.314510 J/(mol K)

Boltzmann's constant: $k = 1.380658\ 10^{-23}$ J K^{-1}

Stefan's constant: σ = 56.703 nW/(m_ K^4)

Proton charge: $e = 1.60217733\ 10^{-19}$ C

Electron mass: $m_e = 9.1093897\ 10^{-31}$ kg

Planck's constant: $h = 6.6260755\ 10^{-34}$ J s

Base of centigrade scale: T_K = 273.15 K
Sun mass: $M_S = 1.991\ 10^{30}$ kg

Earth mass: $M_E = 5.979\ 10^{24}$ kg

Mean radius of Earth: r_E = 6.373 Mm
Major semiaxis of Earth orbit: $R_E = 1.4957\ 10^{11}$ m

Sidereal day: d_S = 86.16406 ks
Year: y = 31.558150 Ms
Standard value of the gravitational field at the Earth surface: g = 9.80665 m s^{-2}

Standard value of the atmospheric pressure at sea level: p_0 = 101325 Pa
Refractive index of air for visibile light, at standard pressure and 15 °C: n_{air} = 1.000277

Solar constant: S = 1355 W m^{-2}

Jupiter mass: $M = 1.901\ 10^{27}$ kg

Equatorial Jupiter radius: R_B = 69.8 Mm
Average radius of Jupiter's orbit: $R = 7.783\ 10^{11}$ m

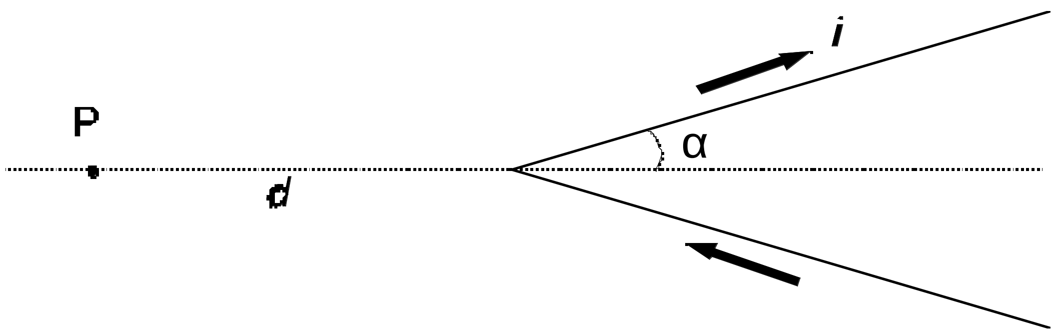
Jovian day: d_J = 35.6 ks
Jovian year: y_J = 374.32 Ms
π: 3.14159265

Problem 2

Magnetic field with a V-shaped wire

Among the first successes of the interpretation by Ampère of magnetic phenomena, we have the computation of the magnetic field **B** *generated by wires carrying an electric current, as compared to early assumptions originally made by Biot and Savart.*

A particularly interesting case is that of a very long thin wire, carrying a constant current i, made out of two rectilinear sections and bent in the form of a "V", with angular half-span[1] α (see figure). According to Ampère's computations, the magnitude B of the magnetic field in a given point P lying on the axis of the "V", outside of it and at a distance d from its vertex, is proportional to $\tan\left(\frac{\alpha}{2}\right)$. Ampère's work was later embodied in Maxwell's electromagnetic theory, and is universally accepted.

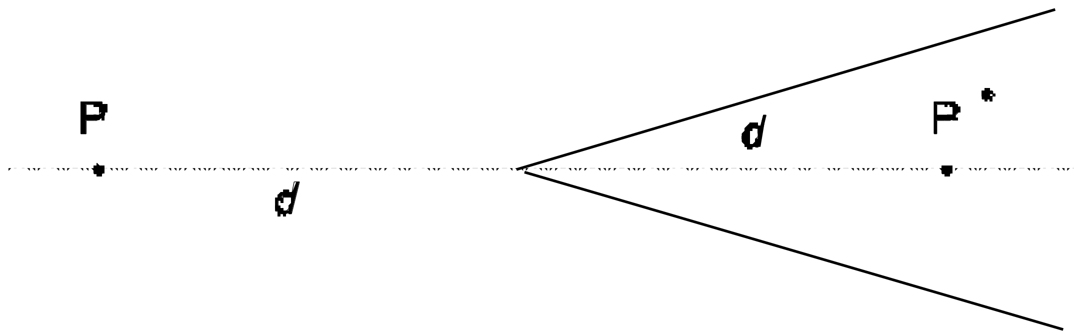


Using our contemporary knowledge of electromagnetism,

1. Find the direction of the field **B** in P. *[1 point]*
2. Knowing that the field is proportional to $\tan\left(\frac{\alpha}{2}\right)$, find the proportionality factor k in $|\mathbf{B}(\mathrm{P})| = k\tan\left(\frac{\alpha}{2}\right)$. *[1.5 points]*
3. Compute the field **B** in a point P^* symmetric to P with respect to the vertex, *i.e.* along the axis and at the same distance d, but inside the "V" (see figure). *[2 points]*

[1] Throughout this problem α is expressed in radians.

Problem 2



4. In order to measure the magnetic field, we place in P a small magnetic needle with moment of inertia I and magnetic dipole moment μ; it oscillates around a fixed point in a plane containing the direction of **B**. Compute the period of small oscillations of this needle as a function of B. *[2.5 points]*

In the same conditions Biot and Savart had instead assumed that the magnetic field in P might have been (we use here the modern notation) $B(\mathrm{P}) = \frac{i\mu_0 \alpha}{\pi^2 d}$, where μ_0 is the magnetic permeability of vacuum. In fact they attempted to decide with an experiment between the two interpretations (Ampère's and Biot and Savart's) by measuring the oscillation period of the magnetic needle as a function of the "V" span. For some α values, however, the differences are too small to be easily measurable.

5. If, in order to distinguish experimentally between the two predictions for the magnetic needle oscillation period T in P, we need a difference by at least 10%, namely $T_1 > 1.10\, T_2$ (T_1 being the Ampere prediction and T_2 the Biot-Savart prediction) state in which range, approximately, we must choose the "V" half-span α for being able to decide between the two interpretations. *[3 points]*

Hint

Depending on which path you follow in your solution, the following trigonometric equation might be useful: $\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1+\cos\alpha}$

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Problem	2
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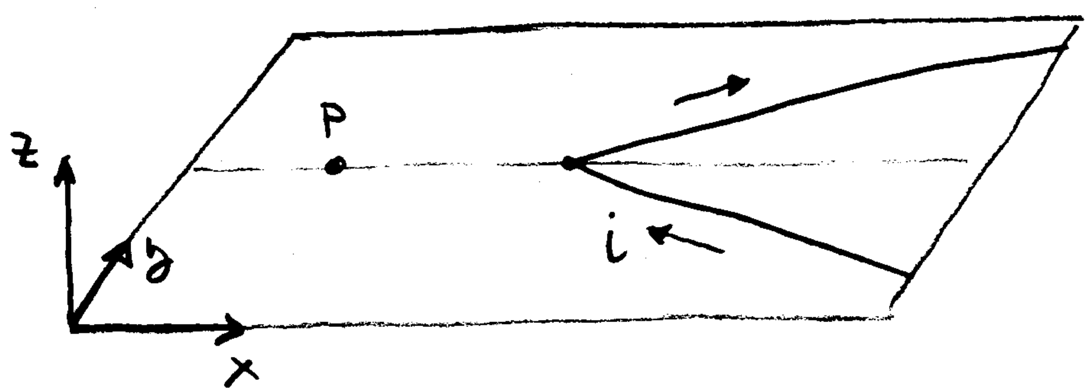
Answer sheet

In this problem write the requested results as analytic expressions, not as numerical values and units, unless explicitly indicated.

1. Using the following sketch draw the direction of the **B** field (the length of the vector is not important). The sketch is a spatial perspective view.



2. Proportionality factor k ……………………….

3. Absolute value of the magnetic field intensity at the point P*, as described in the text………………………..………………

Draw the direction of the **B** field in the above sketch

4. Period of the small angle oscillations of the magnet ……………………………

5. Write for which range of α values (indicating here the numerical values of the range limits) the ratio between the oscillation periods, as predicted by Ampère and by Biot and Savart, is larger than 1.10:

Problem 3

A space probe to Jupiter

We consider in this problem a method frequently used to accelerate space probes in the desired direction. The space probe flies by a planet, and can significantly increase its speed and modify considerably its flight direction, by taking away a very small amount of energy from the planet's orbital motion. We analyze here this effect for a space probe passing near Jupiter.

The planet Jupiter orbits around the Sun along an elliptical trajectory, that can be approximated by a circumference of average radius R; in order to proceed with the analysis of the physical situation we must first:

1. Find the speed V of the planet along its orbit around the Sun. *[1.5 points]*
2. When the probe is between the Sun and Jupiter (on the segment Sun-Jupiter), find the distance from Jupiter where the Sun's gravitational attraction balances that by Jupiter. *[1 point]*

A space probe of mass m = 825 kg flies by Jupiter. For simplicity assume that the trajectory of the space probe is entirely in the plane of Jupiter's orbit; in this way we neglect the important case in which the space probe is expelled from Jupiter's orbital plane.

We only consider what happens in the region where Jupiter's attraction overwhelms all other gravitational forces.

In the reference frame of the Sun's center of mass the initial speed of the space probe is v_0 $=1.00\cdot10^4$ m/s (along the positive y direction) while Jupiter's speed is along the negative x direction (see figure 1); by "initial speed" we mean the space probe speed when it's in the interplanetary space, still far from Jupiter but already in the region where the Sun's attraction is negligible with respect to Jupiter's. We assume that the encounter occurs in a sufficiently short time to allow neglecting the change of direction of Jupiter along its orbit around the Sun. We also assume that the probe passes behind Jupiter, i.e. the x coordinate is greater for the probe than for Jupiter when the y coordinate is the same.

Problem 3

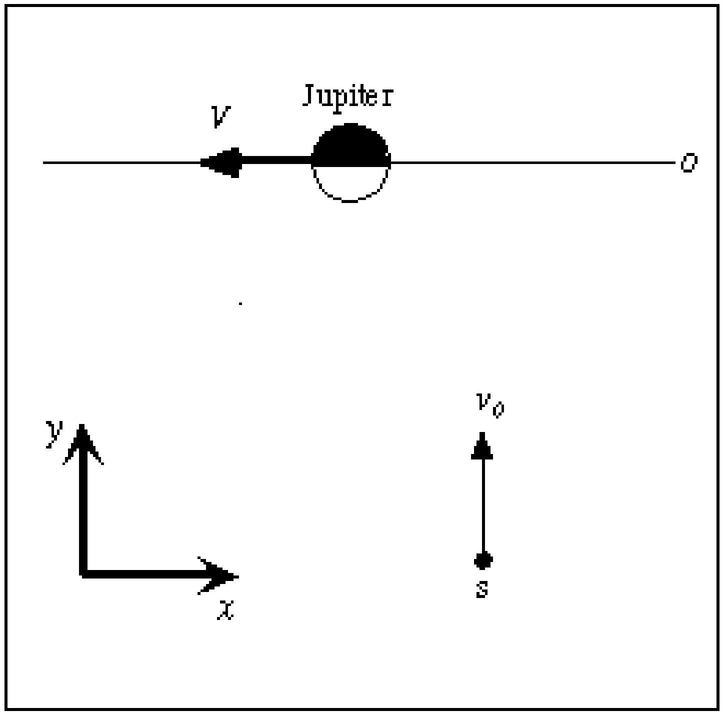


Figure 1: View in the Sun center of mass system. O denotes Jupiter's orbit, s is the space probe.

3. Find the space probe's direction of motion (as the angle φ between its direction and the x axis) and its speed v' in Jupiter's reference frame, when it's still far away from Jupiter. *[2 points]*
4. Find the value of the space probe's total mechanical energy E in Jupiter's reference frame, putting – as usual – equal to zero the value of its potential energy at a very large distance, in this case when it is far enough to move with almost constant velocity owing to the smallness of all gravitational interactions. *[1 point]*

The space probe's trajectory in the reference frame of Jupiter is a hyperbola and its equation in polar coordinates in this reference frame is

$$\frac{1}{r} = \frac{GM_G m_s^2}{l^2}\left(1+\sqrt{1+\frac{2El^2}{G^2M_G^2 m_s^3}}\cos\theta\right) \qquad (1)$$

where b is the distance between one of the asymptotes and Jupiter (the so called *impact parameter*), E is the probe's total mechanical energy in Jupiter's reference frame, G is the gravitational constant, M is the mass of Jupiter, r and θ are the polar coordinates (the radial distance and the polar angle).

Figure 2 shows the two branches of a hyperbola as described by equation (1); the asymptotes and the polar co-ordinates are also shown. Note that equation (1) has its origin in the "attractive focus" of the hyperbola. The space probe's trajectory is the attractive trajectory (the emphasized branch).

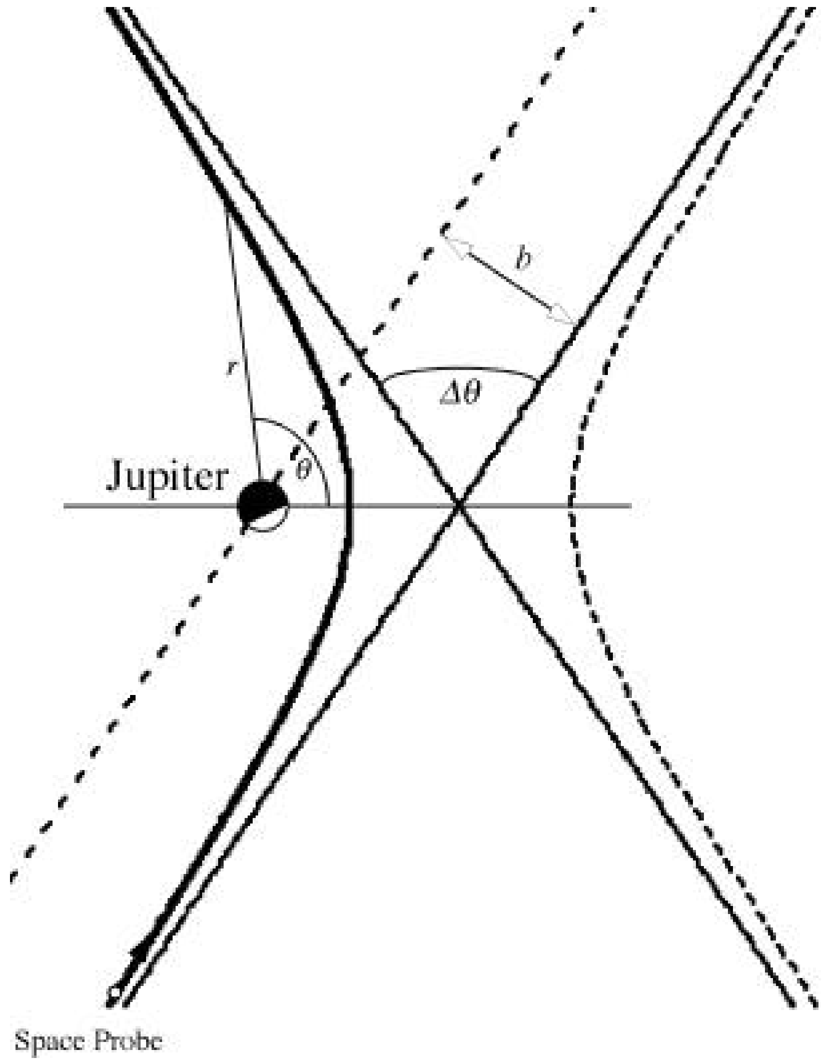


Figure 2

5. Using equation (1) describing the space probe's trajectory, find the total angular deviation $\Delta\theta$ in Jupiter's reference frame (as shown in figure 2) and express it as a function of initial speed v' and impact parameter b. *[2 points]*
6. Assume that the probe cannot pass Jupiter at a distance less than three Jupiter radii from the center of the planet; find the minimum possible impact parameter and the maximum possible angular deviation. *[1 point]*
7. Find an equation for the final speed v'' of the probe in the Sun's reference frame as a function only of Jupiter's speed V, the probe's initial speed v_0 and the deviation angle $\Delta\theta$. *[1 point]*
8. Use the previous result to find the numerical value of the final speed v'' in the Sun's reference frame when the angular deviation has its maximum possible value. *[0.5 points]*

Problem 3

Hint

Depending on which path you follow in your solution, the following trigonometric formulas might be useful:

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

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Answer sheet

Unless explixitly requested to do otherwise, in this problem you are supposed to write your results both as analytic equations (first) and then as numerical results and units (*e.g.* $A=bc=1.23$ m^2).

1. Speed V of Jupiter along its orbit ……………………………

2. Distance from Jupiter where the two gravitational attractions balance each other …………………………………….

3. Initial speed v' of the space probe in Jupiter's reference frame ……………………………….. and the angle ϕ its direction forms with the x axis, as defined in figure 1, …………………..………

4. Total energy E of the space probe in Jupiter's reference frame …………………………………...

5. Write a formula linking the probe's deviation $\Delta\theta$ in Jupiter's reference frame to the impact parameter b, the initial speed v' and other known or computed quantities ……………………………………………………………………………………………..

6. If the distance from Jupiter's center can't be less than three Jovian radii, write the minimum impact parameter and the maximum angular deviation: b = ……………………………………; $\Delta\theta$ = ……………………………………………………..

7. Equation for the final probe speed v'' in the Sun's reference frame as a function of V, v_0 and $\Delta\theta$ …..……………………………..…………………………………………...………

8. Numerical value of the final speed in the Sun's reference frame when the angular deviation has its maximum value as computed in step 6 ………….…………………………………………

Solution

1. At equilibrium the pressure p inside the vessel must be equal to the room pressure p_0 plus the pressure induced by the weight of the movable base: $p = p_0 + \frac{mg}{\pi r^2}$. This is true before and after irradiation. Initially the gas temperature is room temperature. Owing to the state equation of perfect gases, the initial gas volume V_1 is $V_1 = \frac{nRT_0}{p}$ (where R is the gas constant) and therefore the height h_1 of the cylinder which is occupied by the gas is $h_1 = \frac{V_1}{\pi r^2} = \frac{nRT_0}{p_0 \pi r^2 + mg}$. After irradiation, this height becomes $h_2 = h_1 + \Delta s$, and therefore the new temperature is
$T_2 = T_0\left(1 + \frac{\Delta s}{h_1}\right) = T_0 + \frac{\Delta s(p_0 \pi r^2 + mg)}{nR}$.
Numerical values: p = 102.32 kPa; T_2 = 322 K = 49°C

2. The mechanical work made by the gas against the plate weight is $mg\Delta s$ and against the room pressure is $p_0 \pi r^2 \Delta s$, therefore the total work is $L = (mg + p_0 \pi r^2)\Delta s$ = 24.1 J

3. The internal energy, owing to the temperature variation, varies by an amount $\Delta U = nc_V(T_2 - T_0)$. The heat introduced into the system during the irradiation time Δt is $Q = \Delta U + L = nc_V \frac{T_0 \Delta s}{h_1} + (mg + p_0 \pi r^2)\Delta s = \Delta s(p_0 \pi r^2 + mg)\left(\frac{c_V}{R} + 1\right)$. This heat comes exclusively from the absorption of optical radiation and coincides therefore with the absorbed optical energy, Q = 84 J.

   The same result can also be obtained by considering an isobaric transformation and remembering the relationship between molecular heats:
   $$Q = nc_p(T_2 - T_0) = n(c_V + R)\left[\frac{\Delta s(p_0 \pi r^2 + mg)}{nR}\right] = \Delta s(p_0 \pi r^2 + mg)\left(\frac{c_V}{R} + 1\right)$$

4. Since the laser emits a constant power, the absorbed optical power is $W = \frac{Q}{\Delta t} = \left(\frac{c_V}{R} + 1\right)\frac{\Delta s}{\Delta t}(p_0 \pi r^2 + mg)$ = 8.4 W. The energy of each photon is hc/λ, and thus the number of photons absorbed per unit time is $\frac{W\lambda}{hc}$ = $2.2 \cdot 10^{19}$ s^{-1}

5. The potential energy change is equal to the mechanical work made against the plate weight, therefore the efficiency η of the energy transformation is

$$\frac{mg\Delta s}{Q} = \frac{1}{\left(1+\frac{p_0\pi\, r^2}{mg}\right)\left(1+\frac{c_V}{R}\right)} = 2.8\cdot 10^{-3} \approx 0.3\%$$

6. When the cylinder is rotated and its axis becomes horizontal, we have an adiabatic transformation where the pressure changes from p to p_0, and the temperature changes therefore to a new value T_3. The equation of the adiabatic transformation $pV^\gamma = \text{constant}$ may now be written in the form $T_3 = T_2\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}$, where $\gamma = \frac{c_p}{c_V} = \frac{c_V + R}{c_V} = 1 + \frac{R}{c_V} = 1.399$. Finally T_3 = 321 K = 48°C

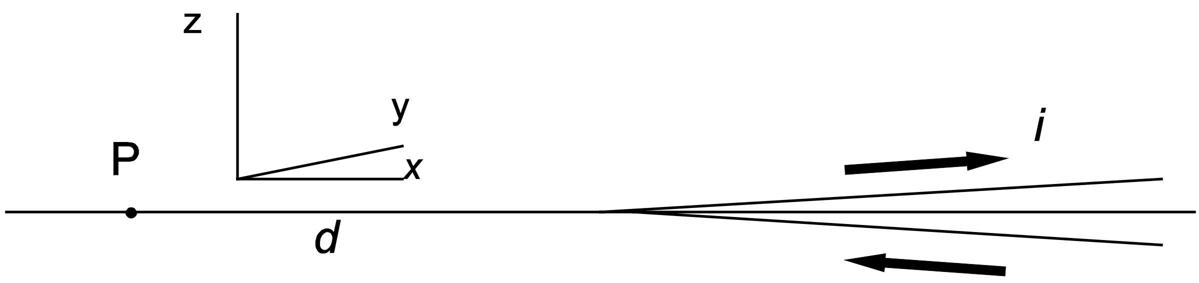
Grading guidelines

1.	0.5	Understanding the relationship between inner and outer pressure
	0.7	Proper use of the plate displacement
	0.2+0.2	Correct results for final pressure
	0.2+0.2	Correct results for final temperature
2.	0.6	Understanding that the work is made both against plate weight and against atmospheric pressure
	0.2+0.2	Correct results for work
3.	1	Correct approach
	0.5	Correct equation for heat
	0.3	Understanding that the absorbed optical energy equals heat
	0.2	Correct numerical result for optical energy
4.	0.2+0.2	Correct results for optical power
	0.5	Einstein's equation
	0.3+0.3	Correct results for number of photons
5.	0.6	Computation of the change in potential energy
	0.2+0.2	Correct results for efficiency
6.	0.8	Understanding that the pressure returns to room value
	0.4	Understanding that there is an adiabatic transformation
	0.4	Equation of adiabatic transformation
	0.5	Derivation of γ from the relationship between specific heats
	0.2+0.2	Correct results for temperature

For "correct results" two possible marks are given: the first one is for the analytical equation and the second one for the numerical value.
For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing.
No bonus can be given for taking into account the gas weight

Solution

1. The contribution to **B** given by each leg of the "V" has the same direction as that of a corresponding infinite wire and therefore - if the current proceeds as indicated by the arrow - the magnetic field is orthogonal to the wire plane taken as the *x-y* plane. If we use a right-handed reference frame as indicated in the figure, **B**(P) is along the positive *z* axis.



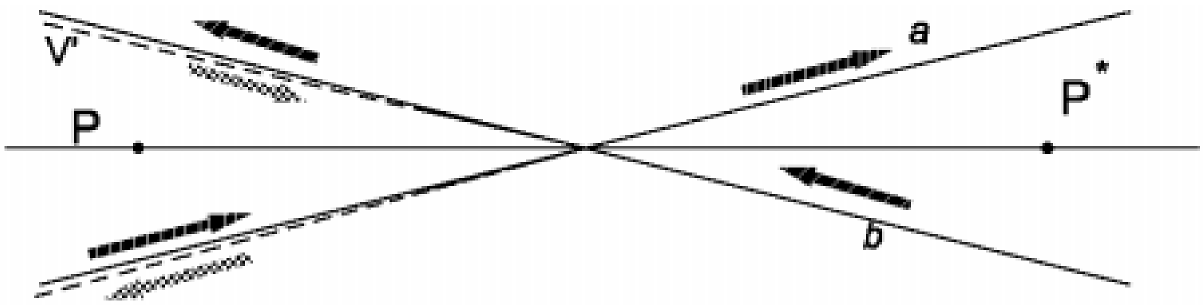
For symmetry reasons, the total field is twice that generated by each leg and has still the same direction.

2A. When $\alpha=\pi/2$ the "V" becomes a straight infinite wire. In this case the magnitude of the field B(P) is known to be $B = \frac{i}{2\pi\varepsilon_0 c^2 d} = \frac{i\mu_0}{2\pi d}$, and since tan($\pi$/4)=1, the factor k is $\frac{i\,\mu_0}{2\pi d}$.

The following solution is equally acceptable:

2B. If the student is aware of the equation $B = \frac{\mu_0 i}{4\pi}\frac{\cos\theta_1 - \cos\theta_2}{h}$ for a finite stretch of wire lying on a straight line at a distance h from point P and whose ends are seen from P under the angles θ_1 and θ_2, he can find that the two legs of the "V" both produce fields $\frac{\mu_0 i}{4\pi}\frac{1-\cos\alpha}{d\,\sin\alpha}$ and therefore the total field is $B = \frac{i\mu_0}{2\pi\, d}\frac{1-\cos\alpha}{\sin\alpha} = \frac{i\mu_0}{2\pi\, d}\tan\left(\frac{\alpha}{2}\right)$. This is a more complete solution since it also proves the angular dependence, but it is not required.

3A. In order to compute **B**(P*) we may consider the "V" as equivalent to two crossed infinite wires (*a* and *b* in the following figure) plus another "V", symmetrical to the first one, shown in the figure as V', carrying the same current *i*, in opposite direction.



Then $B(\mathrm{P}^*) = B_a(\mathrm{P}^*) + B_b(\mathrm{P}^*) + B_{\mathrm{V'}}(\mathrm{P}^*)$. The individual contributions are:

$B_a(\mathrm{P}^*) = B_b(\mathrm{P}^*) = \dfrac{i\mu_0}{2\pi d \sin\alpha}$, along the negative z axis;

$B_{\mathrm{V'}}(\mathrm{P}^*) = \dfrac{i\mu_0}{2\pi d} \tan\left(\dfrac{\alpha}{2}\right)$, along the positive z axis.

Therefore we have $B(\mathrm{P}^*) = \dfrac{i\mu_0}{2\pi d}\left[\dfrac{2}{\sin\alpha} - \tan\left(\dfrac{\alpha}{2}\right)\right] = k\left(\dfrac{1+\cos\alpha}{\sin\alpha}\right) = k\cot\left(\dfrac{\alpha}{2}\right)$, and the field is along the negative z axis.

The following solutions are equally acceptable:

3B. The point P^* inside a "V" with half-span α can be treated as if it would be on the outside of a "V" with half-span π-α carrying the same current but in an opposite way, therefore the field is $B(\mathrm{P}^*) = k\tan\left(\dfrac{\pi-\alpha}{2}\right) = k\tan\left(\dfrac{\pi}{2} - \dfrac{\alpha}{2}\right) = k\cot\left(\dfrac{\alpha}{2}\right)$; the direction is still that of the z axis but it is along the negative axis because the current flows in the opposite way as previously discussed.

3C. If the student follows the procedure outlined under 2B., he/she may also find the field value in P* by the same method.

4. The mechanical moment **M** acting on the magnetic needle placed in point P is given by **M** = $\mu \wedge$ **B** (where the symbol $\wedge$ is used for vector product). If the needle is displaced from its equilibrium position by an angle β small enough to approximate sinβ with β, the angular momentum theorem gives $M = -\mu B\beta = \dfrac{dL}{dt} = I\dfrac{d^2\beta}{dt^2}$, where there is a minus sign because the mechanical momentum is always opposite to the displacement from equilibrium. The period T of the small oscillations is therefore given by $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I}{\mu B}}$.

Writing the differential equation, however, is not required: the student should recognise the same situation as with a harmonic oscillator.

5. If we label with subscript A the computations based on Ampère's interpretation, and with subscript BS those based on the other hypothesis by Biot and Savart, we have

Problem 2 – Solution

$$B_{\text{A}} = \frac{i\mu_0}{2\pi d}\tan(\frac{\alpha}{2}) \qquad B_{\text{BS}} = \frac{i\mu_0}{\pi^2 d}\alpha$$

$$T_{\text{A}} = 2\pi\sqrt{\frac{2\pi Id}{\mu_0 \mu i \tan(\frac{\alpha}{2})}} \qquad T_{\text{BS}} = 2\pi\sqrt{\frac{\pi^2 Id}{\mu_0 \mu i \alpha}}$$

$$\frac{T_{\text{A}}}{T_{\text{BS}}} = \sqrt{\frac{2\alpha}{\pi \tan(\frac{\alpha}{2})}}$$

For $\alpha = \pi/2$ (maximum possible value) $T_{\text{A}} = T_{\text{BS}}$; and for $\alpha \to 0$ $T_{\text{A}} \to \frac{2}{\sqrt{\pi}} T_{\text{BS}} \approx 1.128\, T_{\text{BS}}$. Since within this range $\frac{\tan(\alpha/2)}{\alpha/2}$ is a monotonically growing function of α, $\frac{T_{\text{A}}}{T_{\text{BS}}}$ is a monotonically decreasing function of α; in an experiment it is therefore not possible to distinguish between the two interpretations if the value of α is larger than the value for which $T_{\text{A}} = 1.10\, T_{\text{BS}}$ (10% difference), namely when $\tan\left(\frac{\alpha}{2}\right) = \frac{4}{1.21\pi}\frac{\alpha}{2} = 1.05\frac{\alpha}{2}$. By looking into the trigonometry tables or using a calculator we see that this condition is well approximated when $\alpha/2 = 0.38$ rad; α must therefore be smaller than 0.77 rad ≈ 44°.

A graphical solution of the equation for α is acceptable but somewhat lengthy. A series development, on the contrary, is not acceptable.

Grading guidelines

1. 1 for recognising that each leg gives the same contribution
0.5 for a correct sketch

2. 0.5 for recognising that $\alpha=\pi/2$ for a straight wire, or for knowledge of the equation given in 2B.
0.25 for correct field equation (infinite or finite)
0.25 for value of k

3. 0.7 for recognising that the V is equivalent to two infinite wires plus another V
0.3 for correct field equation for an infinite wire
0.5 for correct result for the intensity of the required field
0.5 for correct field direction

alternatively

0.8 for describing the point as outside a V with π-α half-amplitude and opposite current
0.7 for correct analytic result
0.5 for correct field direction

alternatively

0.5 for correctly using equation under 2B
1 for correct analytic result
0.5 for correct field direction

4. 0.5 for correct equation for mechanical moment **M**
0.5 for doing small angle approximation $\sin\beta \approx \beta$
1 for correct equation of motion, including sign, or for recognizing analogy with harmonic oscillator
0.5 for correct analytic result for T

5. 0.3 for correct formulas of the two periods
0.3 for recognising the limiting values for α
0.4 for correct ratio between the periods
1 for finding the relationship between α and tangent
0.5 for suitable approximate value of α
0.5 for final explicit limiting value of α

For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing

Solution

1A. Assuming – as outlined in the text – that the orbit is circular, and relating the radial acceleration $\frac{V^2}{R}$ to the gravitational field $\frac{GM_S}{R^2}$ (where M_S is the solar mass) we obtain Jupiter's orbital speed $V = \sqrt{\frac{GM_S}{R}} \approx 1.306 \cdot 10^4$ m/s.

The following alternative solution is also acceptable:

1B. Since we treat Jupiter's motion as circular and uniform, $V = \omega R = \frac{2\pi R}{y_J}$, where y_J is the revolution period of Jupiter, which is given in the list of the general physical constants.

2. The two gravitational forces on the space probe are equal when

$$\frac{GMm}{\rho^2} = \frac{GM_S m}{(R-\rho)^2} \qquad (2)$$

(where ρ is the distance from Jupiter and M is Jupiter's mass), whence

$$\sqrt{M}\,(R-\rho) = \rho\sqrt{M_S} \qquad (3)$$

and

$$\rho = \frac{\sqrt{M}}{\sqrt{M_S}+\sqrt{M}} R = 0.02997 R = 2.333 \cdot 10^{10}\,\mathrm{m} \qquad (4)$$

and therefore the two gravitational attractions are equal at a distance of about 23.3 million kilometers from Jupiter (about 334 Jupiter radii).

3. With a simple Galilean transformation we find that the velocity components of the probe in Jupiter's reference frame are

$$\begin{cases} v'_x = V \\ v'_y = v_0 \end{cases}$$

and therefore - in Jupiter's reference frame – the probe travels with an angle $\theta_0 = \arctan\frac{v_0}{V}$ with respect to the x axis and its speed is $v' = \sqrt{v_0^2 + V^2}$ (we also note that $\cos\theta_0 = \frac{V}{\sqrt{v_0^2+V^2}} = \frac{V}{v'}$, and $\sin\theta_0 = \frac{v_0}{\sqrt{v_0^2 + V^2}} = \frac{v_0}{v'}$).

Using the given values we obtain $\theta_0 = 0.653$ rad $\approx 37.4°$ and $v' = 1.65 \cdot 10^4$ m/s .

4. Since the probe trajectory can be described only approximately as the result of a two-body gravitational interaction (we should also take into account the interaction with the Sun and other planets) we assume a large but not infinite distance from Jupiter and we approximate the total energy in Jupiter's reference frame as the probe's kinetic energy at that distance:

$$E \approx \frac{1}{2} m v'^2 \tag{5}$$

The corresponding numerical value is E = 112 GJ.

5. Equation (1) shows that the radial distance becomes infinite, and its reciprocal equals zero, when

$$1 + \sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}} \cos\theta = 0 \tag{7}$$

namely when

$$\cos\theta = -\frac{1}{\sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}} \tag{8}$$

We should also note that the radial distance can't be negative, and therefore its acceptable values are those satisfying the equation

$$1 + \sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}} \cos\theta \geq 0 \tag{9}$$

or

$$\cos\theta \geq -\frac{1}{\sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}} \tag{10}$$

The solutions for the limiting case of eq. (10) (i.e. when the equal sign applies) are:

$$\theta_\pm = \pm \arccos\left[-\left(1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}\right)^{-1/2}\right] = \pm\left(\pi - \arccos\frac{1}{\sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}}\right) \tag{11}$$

and therefore the angle $\Delta\theta$ (shown in figure 2) between the two hyperbola asymptotes is given by:

$$\Delta\theta = (\theta_+ - \theta_-) - \pi$$

$$= \pi - 2\arccos\frac{1}{\sqrt{1+\frac{2Ev'^2 b^2}{G^2M^2m}}} \qquad (12)$$

$$= \pi - 2\arccos\frac{1}{\sqrt{1+\frac{v'^4 b^2}{G^2M^2}}}$$

In the last line, we used the value of the total energy as computed in the previous section.

6. The angular deviation is a monotonically decreasing function of the impact parameter, whence the deviation has a maximum when the impact parameter has a minimum. From the discussion in the previous section we easily see that the point of nearest approach is when $\theta = 0$, and in this case the minimum distance between probe and planet center is easily obtained from eq. (1):

$$r_{\min} = \frac{v'^2 b^2}{GM}\left(1+\sqrt{1+\frac{v'^4 b^2}{G^2M^2}}\right)^{-1} \qquad (13)$$

By inverting equation (13) we obtain the impact parameter

$$b = \sqrt{r_{\min}^2 + \frac{2GM}{v'^2} r_{\min}} \qquad (14)$$

We may note that this result can alternatively be obtained by considering that, due to the conservation of angular momentum, we have

$$L = mv'b = mv'_{\min}\, r_{\min}$$

where we introduced the speed corresponding to the nearest approach. In addition, the conservation of energy gives

$$E = \frac{1}{2}mv'^2 = \frac{1}{2}mv'^2_{\min} - \frac{GMm}{r_{\min}}$$

and by combining these two equations we obtain equation (14) again.

The impact parameter is an increasing function of the distance of nearest approach; therefore, if the probe cannot approach Jupiter's surface by less than two radii (and thus r_{min} = $3R_B$, where R_B is Jupiter's body radius), the minimum acceptable value of the impact parameter is

$$b_{min} = \sqrt{9R_B^2 + \frac{6GM}{v'^2} R_B} \qquad (15)$$

From this equation we finally obtain the maximum possible deviation:

$$\Delta\theta_{max} = \pi - 2\arccos\frac{1}{\sqrt{1+\frac{v'^4 b_{min}^2}{G^2M^2}}} = \pi - 2\arccos\frac{1}{\sqrt{1+\frac{v'^4}{G^2M^2}\left(9R_B^2 + \frac{6GM}{v'^2} R_B\right)}} \qquad (16)$$

and by using the numerical values we computed before we obtain:

b_{min} = $4.90 \cdot 10^8$ m $\approx 7.0\ R_B$ and $\Delta\theta_{max}$ = 1.526 rad ≈ 87.4°

7. The final direction of motion with respect to the x axis in Jupiter's reference frame is given by the initial angle plus the deviation angle, thus $\theta_0 + \Delta\theta$ if the probe passes behind the planet. The final velocity components in Jupiter's reference frame are therefore:

$$\begin{cases} v'_x = v'\cos(\theta_0 + \Delta\theta) \\ v'_y = v'\sin(\theta_0 + \Delta\theta) \end{cases}$$

whereas in the Sun reference frame they are

$$\begin{cases} v''_x = v'\cos(\theta_0 + \Delta\theta) - V \\ v''_y = v'\sin(\theta_0 + \Delta\theta) \end{cases}$$

Therefore the final probe speed in the Sun reference frame is

$$
\begin{aligned}
v'' &= \sqrt{(v'\cos(\theta_0+\Delta\theta)-V)^2+(v'\sin(\theta_0+\Delta\theta))^2} \\
&= \sqrt{v_0^2+2V^2-2v'V\cos(\theta_0+\Delta\theta)} \\
&= \sqrt{v_0^2+2V^2-2v'V\left(\cos\theta_0\cos\Delta\theta-\sin\theta_0\sin\Delta\theta\right)} \qquad (17) \\
&= \sqrt{v_0^2+2V^2-2V\left(V\cos\Delta\theta-v_0\ \sin\Delta\theta\right)} \\
&= \sqrt{v_0\left(v_0+2V\sin\Delta\theta\right)+2V^2\left(1-\cos\Delta\theta\right)}
\end{aligned}
$$

8. Using the value of the maximum possible angular deviation, the numerical result is $v'' = 2.62 \cdot 10^4$ m/s.

Grading guidelines

1.	0.4	Law of gravitation, or law of circular uniform motion
	0.4	Correct approach
	0.4+0.3	Correct results for velocity of Jupiter
2.	0.3	Correct approach
	0.4+0.3	Correct results for distance from Jupiter
3.	1	Correct transformation between reference frames
	0.3+0.2	Correct results for probe speed in Jupiter reference frame
	0.3+0.2	Correct results for probe angle
4.	0.8	Understanding how to handle the potential energy at infinity
	0.2	Numerical result for kinetic energy
5.	0.6	Correct approach
	0.6	Equation for the orientation of the asymptotes
	0.8	Equation for the probe deflection angle
6.	0.3+0.2	Correct results for minimum impact parameter
	0.3+0.2	Correct results for maximum deflection angle
7.	0.5	Equation for velocity components in the Sun reference frame
	0.5	Equation for speed as a function of angular deflection
8.	0.5	Numerical result for final speed

For “correct results” two possible marks are given: the first one is for the analytical equation and the second one for the numerical value.
For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing.

30th International Physics Olympiad

Padua, Italy

Experimental competition

Tuesday, July 20th, 1999

<u>Before attempting to assemble your equipment, read the problem text completely!</u>

Please read this first:

1. The time available is 5 hours for one experiment only.
2. Use only the pen provided.
3. Use only the **front side** of the provided sheets.
4. In addition to "blank" sheets where you may write freely, there is a set of *Answer sheets* where you **must** summarize the results you have obtained. Numerical results must be written with as many digits as appropriate; don't forget the units. Try – whenever possible – to estimate the experimental uncertainties.
5. Please write on the "blank" sheets the results of all your measurements and whatever else you deem important for the solution of the problem, that you wish to be evaluated during the marking process. However, you should use mainly equations, numbers, symbols, graphs, figures, and use *as little text as possible*.
6. **It's absolutely imperative** that you write on top of *each* sheet that you'll use: your name ("**NAME**"), your country ("**TEAM**"), your student code (as shown on your identification tag, "**CODE**"), and additionally on the "blank" sheets: the progressive number of each sheet (from 1 to N, "**Page n.**") and the total number (N) of "blank" sheets that you use and wish to be evaluated ("**Page total**"); leave the "**Problem**" field blank. It is also useful to write the number of the section you are answering at the beginning of each such section. If you use some sheets for notes that you don't wish to be evaluated by the marking team, just put a large cross through the whole sheet, and don't number it.
7. When you've finished, turn in all sheets <u>in proper order</u> (answer sheets first, then used sheets in order, unused sheets and problem text at the bottom) and put them all inside the

envelope where you found them; then leave everything on your desk. You are not allowed to take anything out of the room.

This problem consists of 11 pages (including this one and the answer sheets).

This problem has been prepared by the Scientific Committee of the 30th IPhO, including professors at the Universities of Bologna, Naples, Turin and Trieste.

Torsion pendulum

In this experiment we want to study a relatively complex mechanical system – a torsion pendulum – and investigate its main parameters. When its rotation axis is horizontal it displays a simple example of bifurcation.

Available equipment

1. A torsion pendulum, consisting of an outer body (not longitudinally uniform) and an inner threaded rod, with a stand as shown in figure 1
2. A steel wire with handle
3. A long hexagonal nut that can be screwed onto the pendulum threaded rod (needed only for the last exercise)
4. A ruler and a right triangle template
5. A timer
6. Hexagonal wrenches
7. A3 Millimeter paper sheets.
8. An adjustable clamp
9. Adhesive tape
10. A piece of T-shaped rod

The experimental apparatus is shown in figure 1; it is a torsion pendulum that can oscillate either around a horizontal rotation axis or around a vertical rotation axis. The rotation axis is defined by a short steel wire kept in tension. The pendulum has an inner part that is a threaded rod that may be screwed in and out, and can be fixed in place by means of a small hexagonal lock nut. This threaded rod can **not** be extracted from the pendulum body.

When assembling the apparatus in step 5 the steel wire must pass through the brass blocks and through the hole in the pendulum, then must be locked in place by keeping it stretched: lock it first at one end, then use the handle to put it in tension and lock it at the other end.

Warning: The wire must be put in tension only to guarantee the pendulum stability. It's not necessary to strain it with a force larger than about 30 N. While straining it, don't bend the wire against the stand, because it might break.

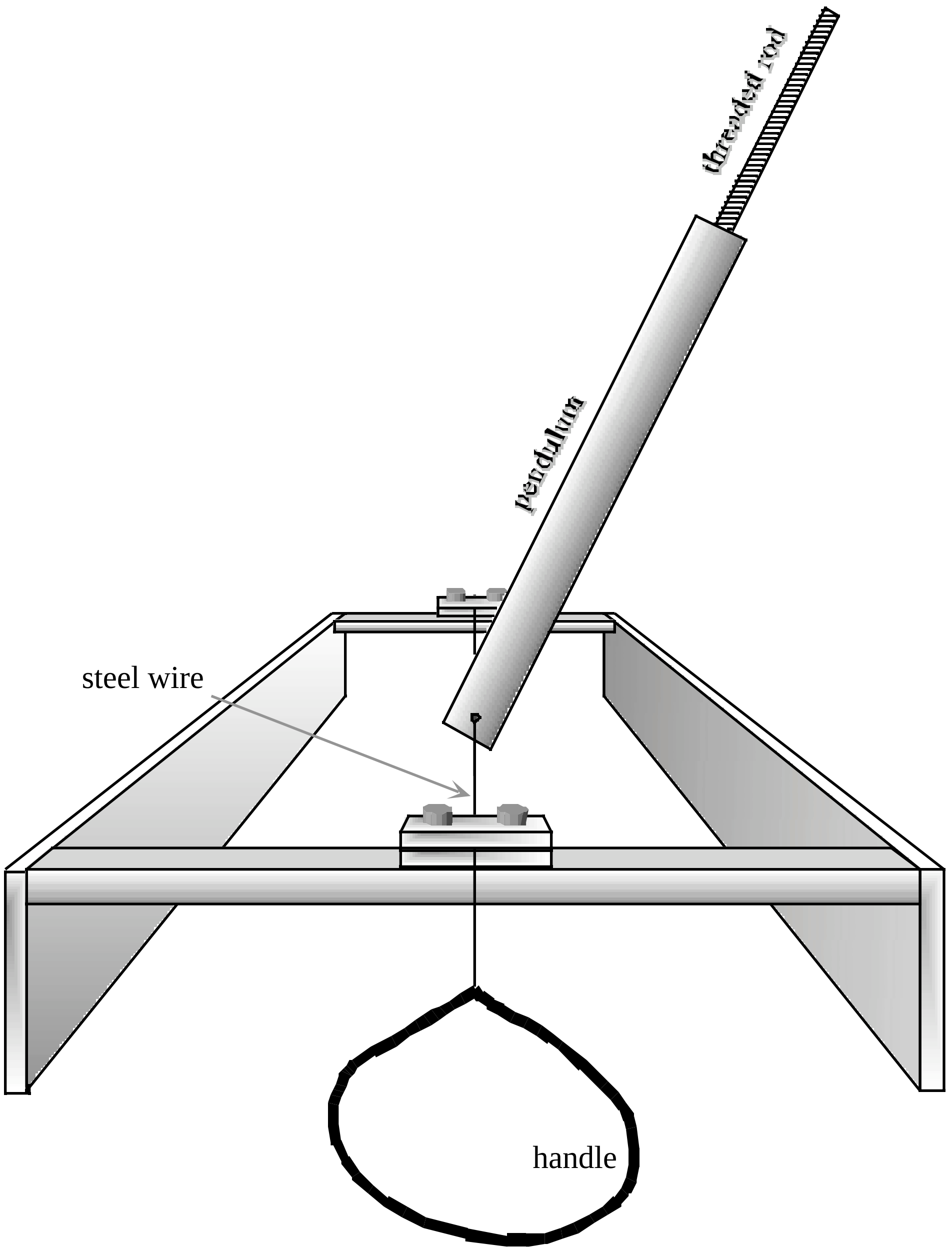


Figure 1: Sketch of the experimental apparatus when its rotation axis is horizontal.

The variables characterizing the pendulum oscillations are:

- the pendulum position defined by the angle θ of deviation from the direction perpendicular to the plane of the stand frame, which is shown horizontal in figure 1.
- the distance x between the free end of the inner threaded rod and the pendulum rotation axis
- the period T of the pendulum oscillations.

The parameters characterizing the system are:

- the torsional elastic constant κ (torque = κ angle) of the steel wire;

- the masses M_1 and M_2 of the two parts of the pendulum (1: outer cylinder[1] and 2: threaded rod);
- the distances R_1 and R_2 of the center of mass of each pendulum part (1: outer cylinder and 2: threaded rod) from the rotation axis. In this case the inner mobile part (the threaded rod) is sufficiently uniform for computing R_2 on the basis of its mass, its length ℓ and the distance x. R_2 is therefore a simple function of the other parameters;
- the moments of inertia I_1 and I_2 of the two pendulum parts (1: outer cylinder and 2: threaded rod). In this case also we assume that the mobile part (the threaded rod) is sufficiently uniform for computing I_2 on the basis of its mass, its length ℓ and the distance x. I_2 is therefore also a simple function of the other parameters;
- the angular position θ_0 (measured between the pendulum and the perpendicular to the plane of the stand frame) where the elastic recall torque is zero. The pendulum is locked to the rotation axis by means of a hex screw, opposite to the threaded rod; therefore θ_0 varies with each installation of the apparatus.

Summing up, the system is described by 7 parameters: κ, M_1, M_2, R_1, I_1,ℓ , θ_0, but θ_0 changes each time the apparatus is assembled, so that only 6 of them are really constants and the purpose of the experiment is that of determining them, namely κ, M_1, M_2, R_1, I_1,ℓ , **experimentally**. Please note that the inner threaded rod can't be drawn out of the pendulum body, and initially only the total mass $M_1 + M_2$ is given (it is printed on each pendulum).

In this experiment several quantities are linear functions of one variable, and you must estimate the parameters of these linear functions. You can use a linear fit, but alternative approaches are also acceptable. The experimental uncertainties of the parameters can be estimated from the procedure of the linear fit or from the spread of experimental data about the fit.

The analysis also requires a simple formula for the moment of inertia of the inner part (we assume that its transverse dimensions are negligible with respect to its length, see figure 2):

$$I_2(x) = \int_{x-\ell}^{x} \lambda\, s^2 ds = \frac{\lambda}{3}\left(x^3 - (x-\ell)^3\right) = \frac{\lambda}{3}\left(3\ell x^2 - 3\ell^2 x + \ell^3\right) \qquad (1)$$

where $\lambda = M_2/\ell$ is the linear mass density, and therefore

$$I_2(x) = M_2 x^2 - M_2 \ell x + \frac{M_2}{3}\ell^2 \qquad (2)$$

[1] Including the small hex locking nut.

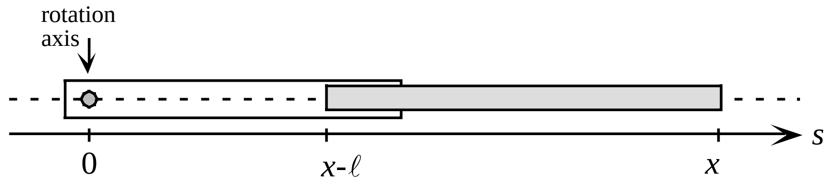


Figure 2: In the analysis of the experiment we can use an equation (eq. 2) for the moment of inertia of a bar whose transverse dimensions are much less than its length. The moment of inertia must be computed about the rotation axis that in this figure crosses the s axis at s=0.

Now follow these steps to find the 6 parameters M_1, M_2, κ, R_1, ℓ , I_1:

1. The value of the total mass M_1+M_2 is given (it is printed on the pendulum), and you can find M_1 and M_2 by measuring the distance *R(x)* between the rotation axis and the center of mass of the pendulum. To accomplish this write first an equation for the position *R(x)* of the center of mass as a function of x and of the parameters M_1, M_2, R_1, ℓ . *[0.5 points]*
2. Now measure *R(x)* for several values of x (at least 3) [2]. Clearly such measurement must be carried out when the pendulum is not attached to the steel wire. Use these measurements and the previous result to find M_1 and M_2. *[3 points]*

---

[2] The small hex nut must be locked in place every time you move the threaded rod. Its mass is included in M_1. This locking must be repeated also in the following, each time you move the threaded rod.

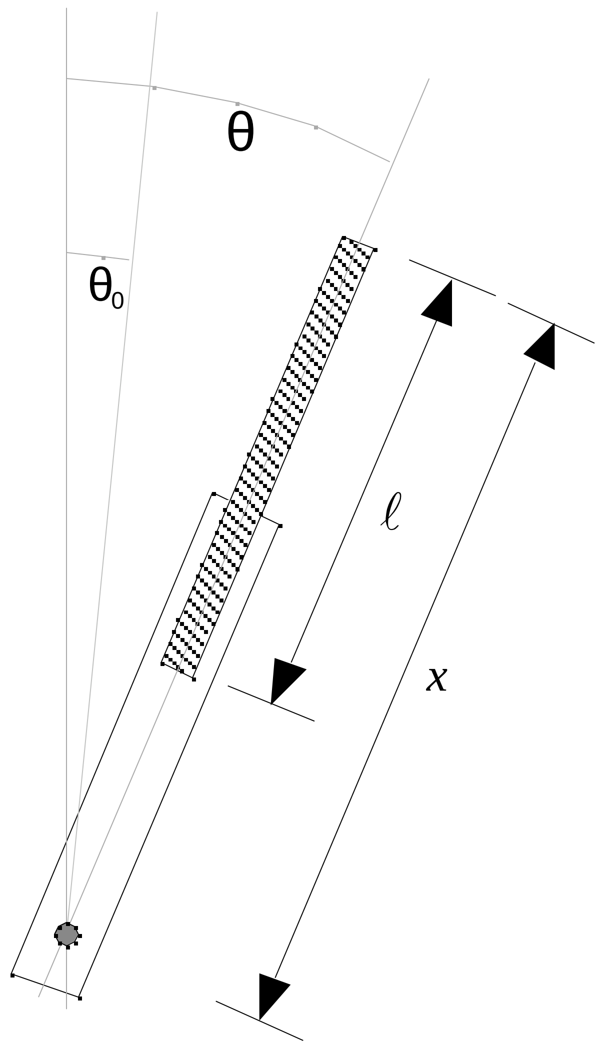


Figure 3: The variables θ and *x* and the parameters θ_0 and ℓ are shown here.

3. Find an equation for the pendulum total moment of inertia *I* as a function of *x* and of the parameters M_2, I_1 and ℓ . *[0.5 points]*
4. Write the pendulum equation of motion in the case of a horizontal rotation axis, as a function of the angle θ (see figure 3) and of *x*, κ, θ_0, M_1, M_2, the total moment of inertia *I* and the position *R(x)* of the center of mass. *[1 point]*
5. In order to determine κ, assemble now the pendulum and set it with its rotation axis horizontal. The threaded rod must initially be as far as possible inside the pendulum. Lock the pendulum to the steel wire, with the hex screw, at about half way between the wire clamps and in such a way that its equilibrium angle (under the combined action of weight and elastic recall) deviates sizeably from the vertical (see figure 4). Measure the equilibrium angle θ_e for several values of *x* (at least 5). *[4 points]*

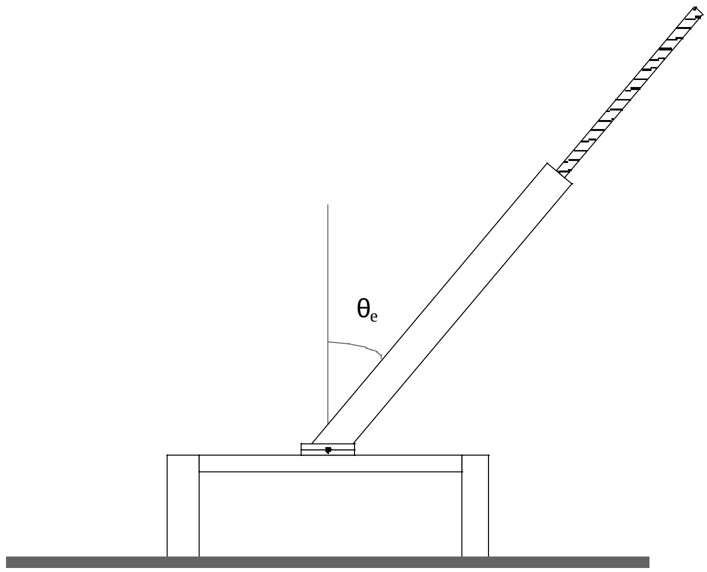


Figure 4: In this measurement set the pendulum so that its equilibrium position deviates from the vertical.

6. Using the last measurements, find κ. *[4.5 points]*
7. Now place the pendulum with its rotation axis vertical[3], and measure its oscillation period for several values of x (at least 5). With these measurements, find I_1 and ℓ. *[4 points]*

At this stage, after having found the system parameters, set the experimental apparatus as follows:

- pendulum rotation axis horizontal
- threaded rod as far as possible inside the pendulum
- pendulum as vertical as possible near equilibrium
- finally add the long hexagonal nut at the end of the threaded rod by screwing it a few turns (it can't go further than that)

In this way the pendulum may have two equilibrium positions, and the situation varies according to the position of the threaded rod, as you can also see from the generic graph shown in figure 5, of the potential energy as a function of the angle θ.

The doubling of the potential energy minimum in figure 5 illustrates a phenomenon known in mathematics as *bifurcation*; it is also related to the various kinds of symmetry breaking that are studied in particle physics and statistical mechanics.

[3] In order to stabilize it in this position, you may reposition the stand brackets.

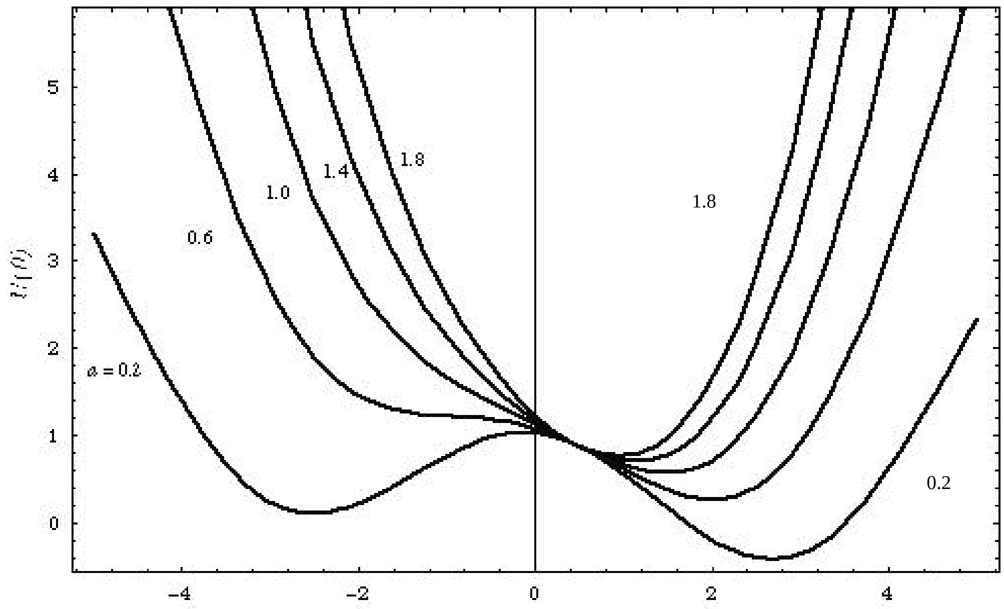


Figure 5: Graph of the function $U(\theta) = \frac{a}{2}(\theta - \theta_0)^2 + \cos\theta$ (which is proportional to the potential energy of this problem) as a function of θ, with $\theta_0 \neq 0$. The various curves correspond to different *a* values as labeled in the figure; smaller values of *a* ($a < 1$) correspond to the appearence of the bifurcation. In our case the parameter *a* is associated with the position *x* of the threaded rod.

We can now study this bifurcation by measuring the period of the small oscillations about the equilibrium position:

8. Plot the period[4] *T* as a function of *x*. What kind of function is it? Is it increasing, decreasing or is it a more complex function? *[2.5 points]*

---

[4] You may be able to observe two equilibrium positions, but one of them is more stable than the other (see figure 5). Report and plot the period for the more stable one.

Solution

The numerical values given in the text are those obtained in a preliminary test performed by a student of the University of Bologna[1], and are reported here only as a guide to the evaluation of the student solutions.

1. and 2. The distance from the center of mass to the rotation axis is:

$$R(x) = \frac{M_1 R_1 + M_2 (x - \ell/2)}{M_1 + M_2} \qquad (1)$$

and therefore, if we measure the position of the center of mass[2] as a function of x we obtain a relationship between the system parameters, and by a linear fit of eq. (1) we obtain an angular coefficient equal to $M_2/(M_1 + M_2)$, and from these equations, making use of the given total mass $M_1 + M_2 =$ 41.0 g ± 0.1 g, we obtain M_1 and M_2. The following table shows some results obtained in the test run.

n	*x* [mm]	*R(x)* [mm]
1	204±1	76±1
2	220±1	83±1
3	236±1	89±1
4	254±1	95±1
5	269±1	101±1
6	287±1	107±1
7	302±1	113±1
8	321±1	119±1

Figure 6 shows the data concerning the position of the pendulum's center of mass together with a best fit straight line: the estimated error on the length measurements is now 1 mm and we treat it as a Gaussian error. Notice that both the dependent variable $R(x)$ and the independent variable x are affected by the experimental uncertainty, however we decide to neglect the uncertainty on x, since it is smaller than 1%. The coefficients a and b in $R(x) = ax+b$ are

$a = 0.366 \pm 0.009$
$b = 2$ mm ± 2 mm

[1] Mr. Maurizio Recchi.
[2] This can easily be done by balancing the pendulum, e.g. on the T-shaped rod provided.

(therefore b is compatible with 0)

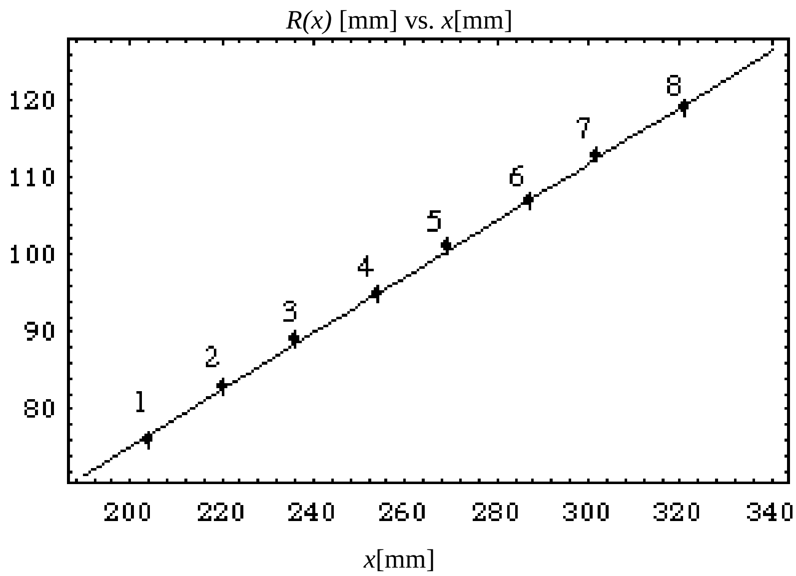


Figure 6: Graph of the position of the pendulum's center of mass (with respect to the rotation axis) as a function of the variable *x*. The numbering of the data points corresponds to that mentioned in the main text. The estimated error is compatible with the fluctuations of the measured data.

For computing the masses only the a value is needed; using the total pendulum mass we find:

M_1 = 26.1 ± 0.4 g
M_2 = 15.0 g ± 0.4 g

Even though many non-programmable pocket calculators can carry out a linear regression, it is likely that many students will be unable to do such an analysis, and in particular they may be unable to estimate the uncertainty of the fit parameters even if their pocket calculators provide a linear regression mode. It is also acceptable to find a and b using several pairs of measurements and finally computing a weighted average of the results. For each pair of measurements a and b are given by

$$a = \frac{y_2 - y_1}{x_2 - x_1} \qquad b = y_2 - ax_2 \qquad (2)$$

and the parameter uncertainties (assuming them gaussian) by

$$\Delta a = a\sqrt{\frac{\Delta x_1^2 + \Delta x_2^2}{(x_1 - x_2)^2} + \frac{\Delta y_1^2 + \Delta y_2^2}{(y_1 - y_2)^2}}$$

$$\Delta b = \sqrt{\Delta y_2^2 + a^2 x_2^2\left(\frac{\Delta x_2^2}{x_2^2} + \frac{\Delta a^2}{a^2}\right)} \qquad (3)$$

In order to calculate (2) and (3) the data can be paired with a scheme like {1,5},{2,6},{3,7},{4,8}, where "far" points are coupled in order to minimize the error on each pair.
There may be other alternative and equally acceptable approaches: they should all be considered valid if the order of magnitude of the estimated uncertainty is correct.

3. The pendulum's total moment of inertia is the sum of the moments of its two parts, and from figure 3 we see that

$$I(x) = I_1 + I_2(x) = M_2 x^2 - M_2 \ell x + \left(I_1 + \frac{M_2}{3}\ell^2\right) \qquad (4)$$

4. The pendulum's equation of motion is

$$I(x)\frac{d^2\theta}{dt^2} = -\kappa(\theta - \theta_0) \qquad (5)$$

if the rotation axis is vertical, while it's

$$I(x)\frac{d^2\theta}{dt^2} = -\kappa(\theta - \theta_0) + (M_1 + M_2)gR(x)sin\theta \qquad (6)$$

if the rotation axis is horizontal.

5. and 6. When the system is at rest in an equilibrium position, the angular acceleration is zero and therefore the equilibrium positions θ_e can be found by solving the equation

$$-\kappa(\theta_\mathrm{e} - \theta_0) + (M_1 + M_2)gR(x)\sin\theta_\mathrm{e} = 0 \qquad (7)$$

If the value x_i corresponds to the equilibrium angle $\theta_{e,i}$, and if we define the quantity (that can be computed from the experimental data) $y_i = (M_1 + M_2)gR(x_i)\sin\theta_{e,i}$, then eq. (7) may be written as

$$y_i = \kappa\theta_{e,i} - \kappa\theta_0 \qquad (8)$$

and therefore the quantities κ and $\kappa\theta_0$ can be found with a linear fit. The following table shows several data collected in a trial run according to the geometry shown in figure 7.

n	*x* [mm]	*h* [mm]	$\sin\theta_e = h/x$	θ_e	*y* [N·μm]
1	204±1	40±1	0.196±0.005	0.197±0.005	6.1±0.3
2	220±1	62±1	0.282±0.005	0.286±0.005	9.4±0.4
3	238±1	75±1	0.315±0.004	0.321±0.005	11.3±0.5
4	255±1	89±1	0.349±0.004	0.357±0.004	13.4±0.5
5	270±1	109±1	0.404±0.004	0.416±0.004	16.4±0.6
6	286±1	131±1	0.458±0.004	0.476±0.004	19.7±0.7
7	307±1	162±1	0.528±0.004	0.556±0.004	24.3±0.8
8	321±1	188±1	0.586±0.004	0.626±0.004	28.2±0.9

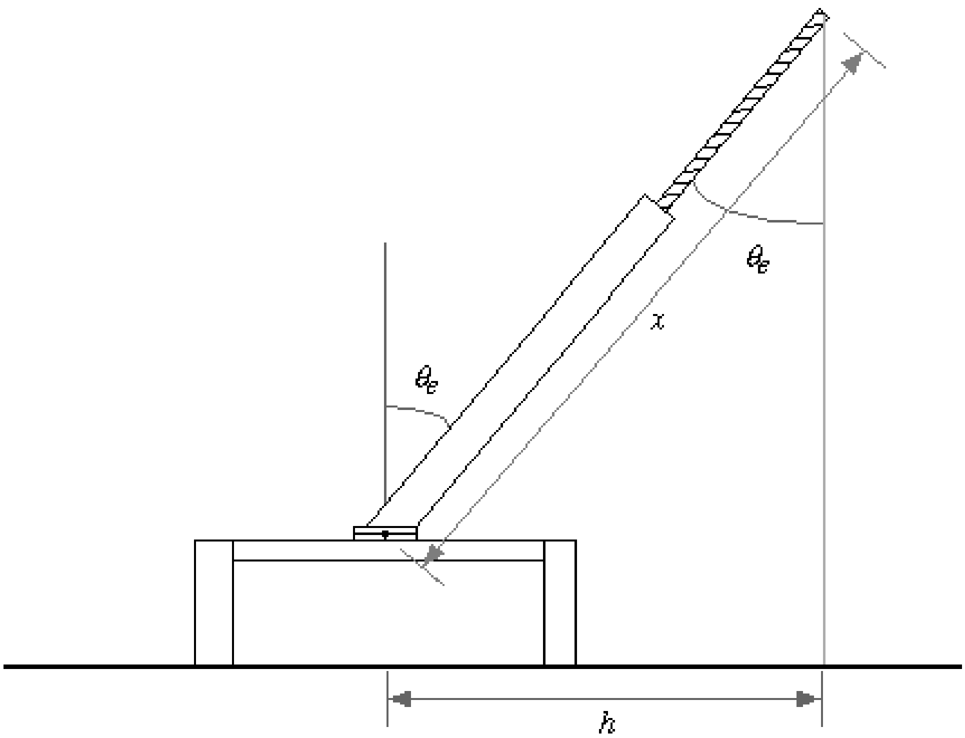


Figure 7: Geometry of the measurements taken for finding the angle.

We see that not only the dependent but also the independent variable is affected by a measurement uncertainty, but the relative uncertainty on θ_e is much smaller than the relative uncertainty on *y* and we neglect it. We obtain from such data (neglecting the first data point, see figure 8):

$\kappa = 0.055$ N·m·rad^{-1} ± 0.001 N·m·rad^{-1}

$\kappa\theta_0$ = -0.0063 N·m ± 0.0008 N·m

Clearly in this case only the determination of the torsion coefficient κ is interesting. The fit of the experimental data is shown in figure 8.

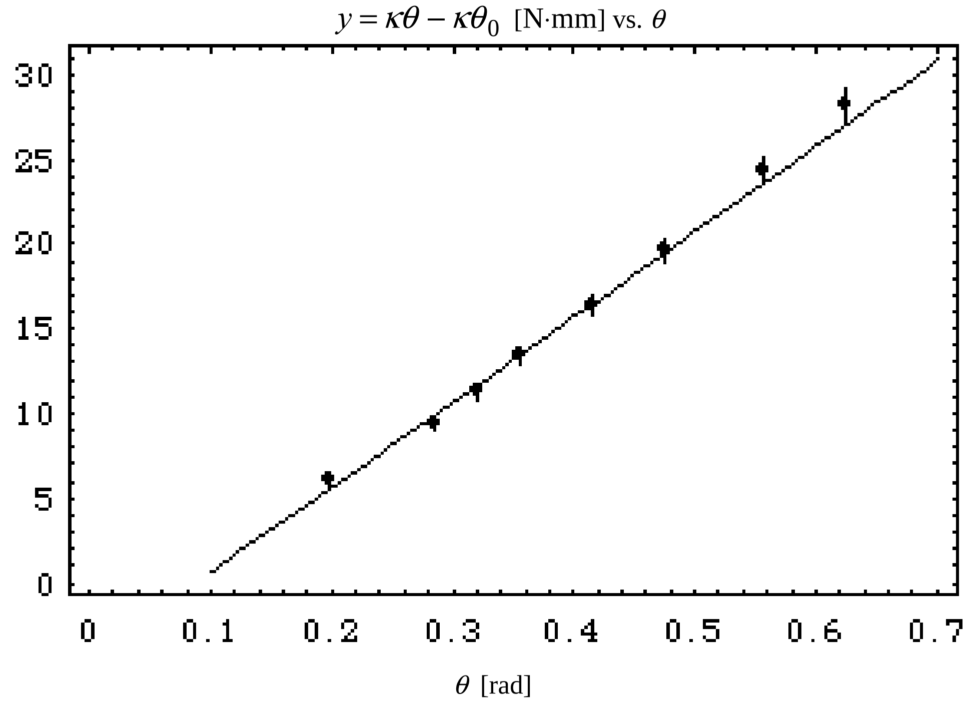


Figure 8: Fit of eq. (8) as a function of θ. In this case the estimated error is again compatible with the experimental data fluctuations. However the data points show a visible deviation from straightness which may be due to an error in the first measurement (the one at lowest θ).

7. The moment of inertia can be found experimentally using the pendulum with its rotation axis vertical and recalling eq. (5); from this equation we see that the pendulum oscillates with angular frequency $\omega(x) = \sqrt{\dfrac{\kappa}{I(x)}}$ and therefore

$$I(x) = \frac{\kappa T^2(x)}{4\pi^2} \qquad (9)$$

where T is the measured oscillation period. Using eq. (9) we see that eq. (4) can be rewritten as

$$\frac{\kappa}{4\pi^2} T^2(x) - M_2 x^2 = -M_2 \ell x + \left(I_1 + \frac{M_2}{3} \ell^2 \right) \qquad (10)$$

The left-hand side in eq. (10) is known experimentally, and therefore with a simple linear fit we can find the coefficients $M_2\ell$ and $\left(I_1 + \frac{M_2}{3}\ell^2\right)$, as we did before. The experimental data are in this case:

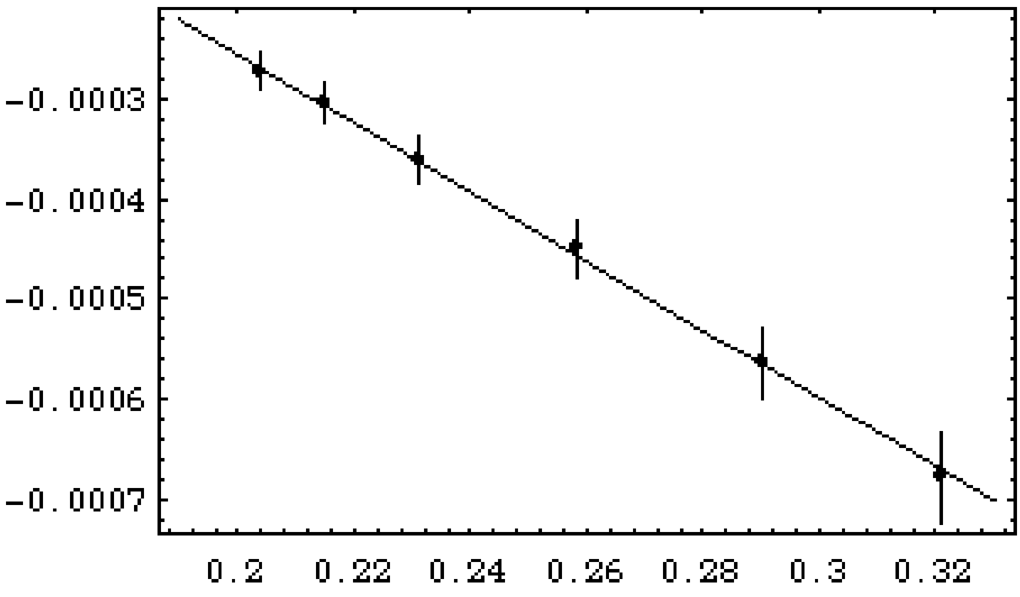
n	x [mm]	T [s]
1	204±1	0.502±0.002
2	215±1	0.528±0.002
3	231±1	0.562±0.002
4	258±1	0.628±0.002
5	290±1	0.708±0.002
6	321±1	0.790±0.002

The low uncertainty on T has been obtained measuring the total time required for 50 full periods.
Using the previous data and another linear fit, we find

ℓ = 230 mm ± 20 mm
$I_1 = 1.7\cdot10^{-4}\ \text{kg}\cdot\text{m}^2 \pm 0.7\cdot10^{-4}\ \text{kg}\cdot\text{m}^2$

and the fit of the experimental data is shown in figure 9.

$y = \frac{\kappa}{4\pi^2}T^2(x) \quad - M_2 x^2$ [kg·m²] vs. x [m]



x [m]

Figure 9: Fit of eq. (10) as a function of x. In this case the estimated error is again compatible with the experimental data fluctuations.

8. Although in this case the period T is a complicated function of x, its graph is simple, and it is shown in figure 10, along with the test experimental data.

The required answer is that there is a single local maximum.

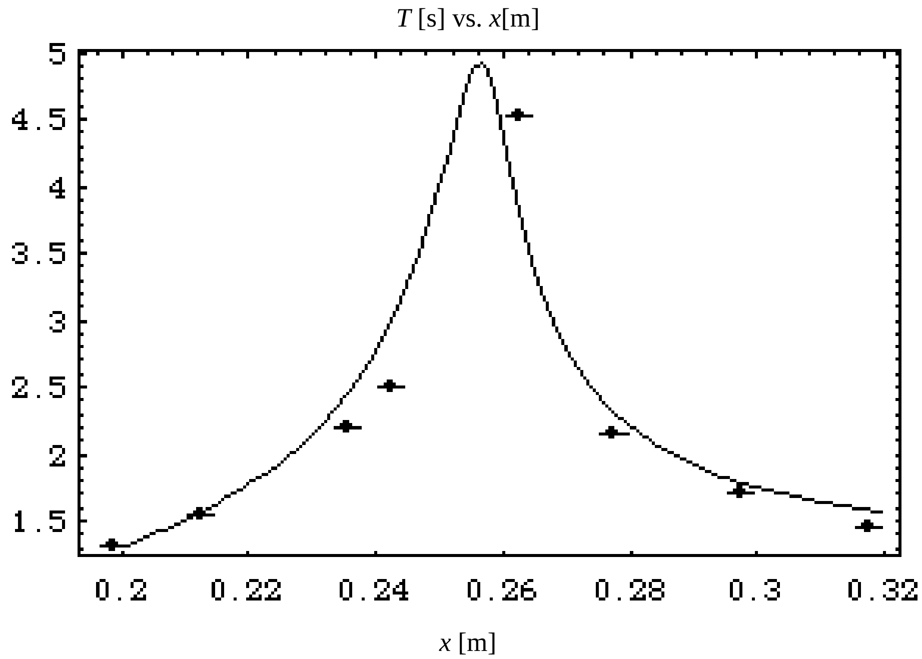


Figure 10: The period T of the pendulum with horizontal axis as a function of x. In addition to the experimental points the figure shows the result of a theoretical calculation of the period in which the following values have been assumed: g = 9.81 m/s^2; κ = 0.056 N·m/rad; M_1 = 0.0261 kg; M_2 = 0.0150 kg; M_3 = 0.00664 kg; $I_1 = 1.0 \cdot 10^{-4}$ kg·m^2; ℓ = 0.21 m; ℓ_3 = 0.025 m; a = 0.365; b = 0.0022 m (so that the position of the center of mass - excluding the final nut of length ℓ_3 - is $R(x) = ax+b$); these are the central measured values, with the exception of κ, I_1 and ℓ which are taken one standard deviation off their central value. Also, the value θ_0 = 0.030 rad ≈ 1.7° has been assumed. Even though the theoretical curve is the result of just a few trial calculations using the measured values (± one standard deviation) and is not a true fit, it is quite close to the measured data.

30th International Physics Olympiad

Padua, Italy

Experimental competition

Comments on the experimental problem.

As mentioned in the problem text, the pendulum may have two equilibrium positions, and the situation varies according to the position of the threaded rod, as shown in figure 5 in the text. The doubling of the potential energy minimum in figure 5 illustrates a phenomenon known in mathematics as bifurcation; it is also related to the various kinds of symmetry breaking that are studied in particle physics and statistical mechanics. It is unlikely that the students will be able to study — other than experimentally — the oscillation period near the bifurcation. Nevertheless, within this discussion, we shall here briefly outline the theoretical point of view.

In order to analyze the peculiar behaviour in the vicinity of the bifurcation, let's work out the mathematics: in general the restoring force is proportional to the angle (θ-θ_0), where θ is the angle between the pendulum and the normal to the plane of the stand frame, and θ_0 is a constant angle, therefore the pendulum's equation of motion is

$$I(x)\frac{d^2\theta}{dt^2} = -\kappa(\theta-\theta_0) \qquad (1)$$

if the rotation axis is vertical, while it is

$$I(x)\frac{d^2\theta}{dt^2} = -\kappa(\theta-\theta_0) + (M_1+M_2)gR(x)\sin\theta \qquad (2)$$

if the rotation axis is horizontal. One can define a potential energy which is a function of the angle θ, and the corresponding formulas for this potential energy are

$$U(\theta;x) = \frac{1}{2}\kappa(\theta-\theta_0)^2 \qquad (3)$$

for a vertical rotation axis and

$$U(\theta;x) = \frac{1}{2}\kappa(\theta-\theta_0)^2 + (M_1+M_2)gR(x)\cos\theta \qquad (4)$$

for a horizontal axis. A graph of eq. (4) is shown in figure 5 in the problem text.

As x increases, the term corresponding to the cosine in the potential energy function (4) becomes more important; at first we have a single energy minimum; then the minimum is displaced and further it separates into two different minima; in general one of them is deeper than the other one. In the most general case, it's possible to have several minima (more than two) but in practice this can't be obtained with the mechanical model in this experiment.

After the addition of mass M_3 whose center of mass is at a distance x_3 from the axis, equation (4) becomes:

$$U(\theta;x)=\frac{1}{2}\kappa(\theta-\theta_0)^2+\left[(M_1+M_2)R(x)+M_3x_3\right]g\cos\theta \qquad (5)$$

From now on, for sake of brevity, we shall write $\alpha(x)=g\left[(M_1+M_2)R(x)+M_3x_3\right]$.

For a quantitative understanding of the bifurcation due to this potential energy, let's consider a simplified equation where we replace the cosine by its series expansion up to the fourth order in θ:

$$U(\theta,x)=\frac{1}{2}\kappa\theta^2+\alpha(x)\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{24}\right) \qquad (6)$$

where we have put θ_0 =0 (see figure D1 for a comparison of the two functions).

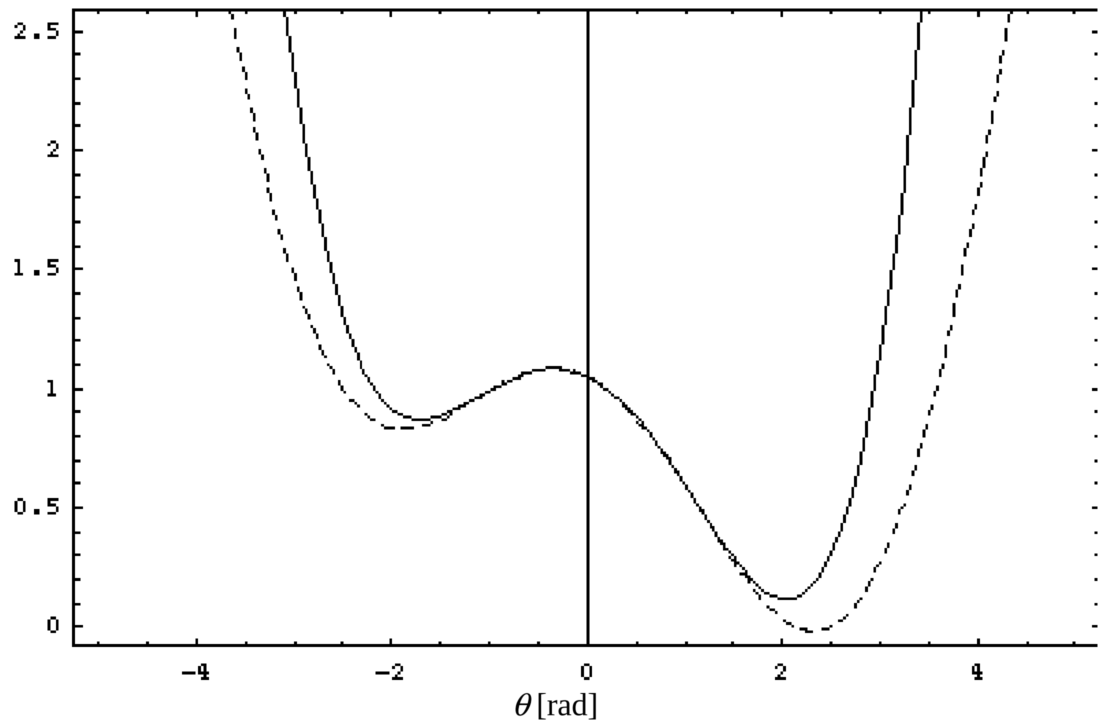


Figure D1: Graph of the functions $(a/2)(\theta-\theta_0)^2+\cos\theta$ (solid line) and $(a/2)(\theta-\theta_0)^2+(1-\theta^2/2+\theta^4/2)$ (dashed line), with a=0.4, θ_0=0.5. The two functions are slightly different, but the minima structure is the same.

Then the first and second derivatives are:

$$\frac{dU}{d\theta} = \kappa\theta + \alpha(x)\left(-\theta + \frac{\theta^3}{6}\right)$$
$$\frac{d^2U}{d\theta^2} = \kappa + \alpha(x)\left(-1 + \frac{\theta^2}{2}\right) \quad (7)$$

We see that when $\alpha(x)$ increases, the second derivative at the origin passes from positive to negative values (that means that we pass at the origin from a situation of a stable minimum to an unstable maximum). The equilibrium position can be found as usual by equating the first derivative (with respect to θ) to zero:

$$\theta\left[\kappa + \alpha(x)\left(-1 + \frac{\theta^2}{6}\right)\right] = 0 \quad (8)$$

and this equation has a solution at the origin (but we already know that at the onset of the bifurcation the origin becomes an unstable maximum) and two other solutions

$$\theta_{\pm} = \pm\sqrt{6\left(1 - \frac{\kappa}{\alpha(x)}\right)} \quad (9)$$

(these angles are imaginary before the bifurcation so that they do not represent physical solutions). Let's now go back to the equation of motion:

$$I_3(x)\frac{d^2\theta}{dt^2} = -\frac{dU}{d\theta} = -\kappa\theta - \alpha(x)\left(-\theta + \frac{\theta^3}{6}\right) \quad (10)$$

where $I_3(x)$ is the total moment of inertia after including the additional mass; without bifurcation we neglect the cubic term and we find that near the origin

$$\frac{d^2\theta}{dt^2} = -\frac{(\kappa - \alpha(x))}{I_3(x)}\theta \quad (11)$$

and therefore the angular frequency of the small oscillations in the (single) stable potential energy minimum without bifurcation is given by

$$\omega^2 = \frac{(\kappa - \alpha(x))}{I_3(x)} \quad (12)$$

and it equals zero at the bifurcation itself, whereas after the onset of the bifurcation the equation of motion becomes

$$\frac{d^2\theta}{dt^2} = -\frac{2(\alpha(x) - \kappa)}{I_3(x)}(\theta - \theta_{\pm}) \quad (13)$$

and therefore the angular frequency of the small oscillations in each stable minimum is given by

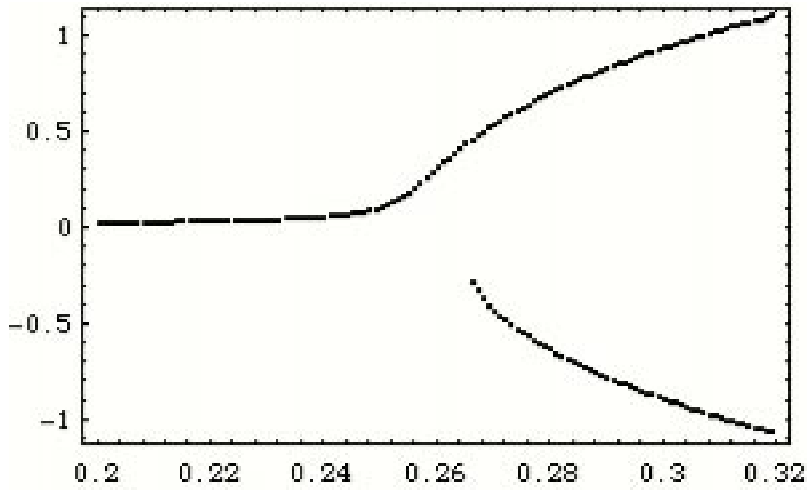
$$\omega^2 = \frac{2(\alpha(x) - \kappa)}{I_3(x)} \tag{14}$$

and it also equals zero at the bifurcation value of x. Since the period is given by $T = 2\pi/\omega$, we can compute it with equations (12) and (14).

If the angle θ_0 is not zero, the computations are considerably more complicated, and can only be performed numerically (some results are show in figures D2 and D3).

The oscillation period has a local maximum near the onset of the bifurcation: the shape of this maximum does not change very much for different misalignment angles θ_0, but the peak value is lower for greater misalignments.

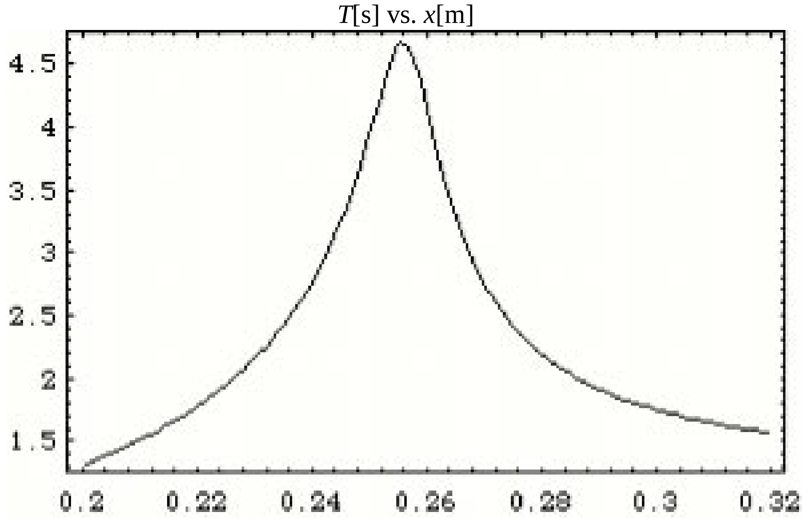
θ [rad] vs. x[m]



x[m]

Figure D2: This figure shows the result of a numerical calculation of the stable minima of the pendulum performed using the data measured in a test run and a misalignement angle θ_0 = 0.0035 rad. The onset of the bifurcation is at $x \approx 0.266$ m.

T[s] vs. x[m]



x[m]

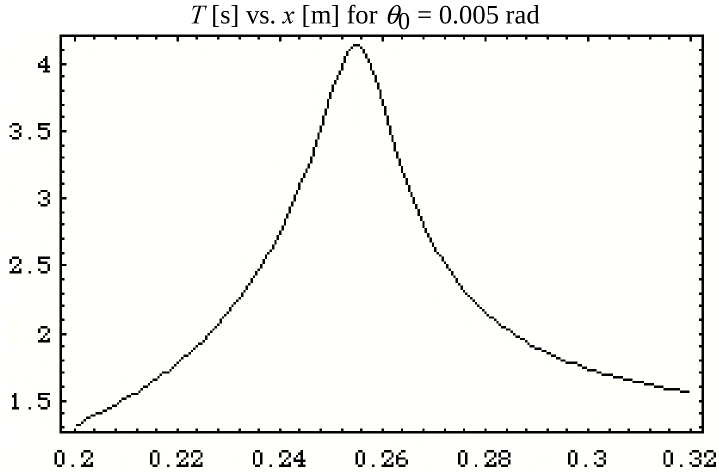
Figure D3: This is a plot of the period T computed with the same data as figure D2. Notice that the period has a local maximum while there is still just one stable position.

The shape of the oscillation time peak is influenced by many parameters, but it is especially sensitive to the angle θ_0. Here are some example plots, calculated with the same data as figures D2 and D3, namely:

g = 9.81 m/s^2;
κ = 0.056 J;
M_1 = 0.0261 kg;
M_2 = 0.0150 kg;
M_3 = 0.00664 kg; mass of the final long nut
$I_1 = 1.\cdot 10^{-4}$ kg·m^2;
ℓ = 0.21 m;
ℓ_3 = 0.025 m; length of the final long nut
a = 0.365;
b = 0.0022 m;
$R(x) = a\cdot x+b$;
$I_3(x) = I_1+M_2\cdot (x^2-\ell\cdot x+\ell^2/3)+(M_3/(3\cdot\ell_3)) \cdot ((x+\ell_3/2)^3-(x-\ell_3/2)^3)$;

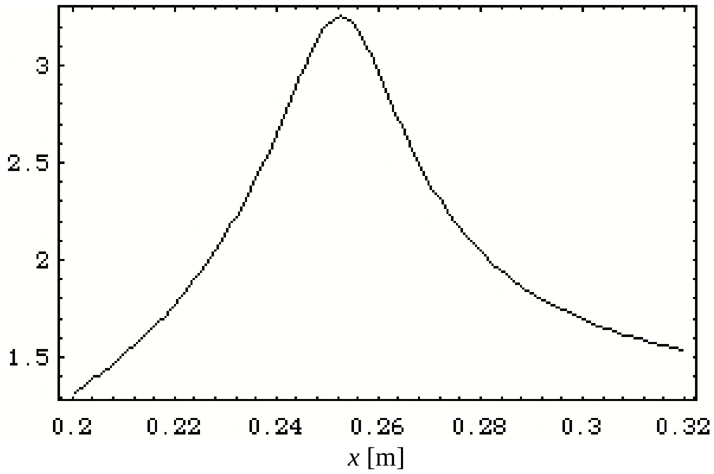
Since it is explicitly requested that the pendulum be as vertical as possible near equilibrium, the shape of the final plot may be used to estimate the experimental prowess of each participant.

T [s] vs. x [m] for θ_0 = 0.005 rad



x [m]

T [s] vs. x [m] for θ_0 = 0.01 rad



T [s] vs. x [m] for θ_0 = 0.03 rad