VERJ. USTNI

k.z.v.š.
CLI

Tra

$F(x,y)$, $F_X(x)$, $F_Y(y)$ naj bodo skupna por. fu. in marginalni por. fu. sluč. vekt. (X,Y). $G(x,y)=\min\{F_X(x), F_Y(y)\}$. ~~naj~~ ~~Izkaže~~ se, da je tudi $G(x,y)$ por. fu. nekega sluč. vektorja.

Pokaži: $\underline{F(x,y) \leq G(x,y), \forall x,y \in \mathbb{R}}$

$F(x,y)$ por. fu. $\Rightarrow$ monotono naraščajoča

$$\Rightarrow \underbrace{\lim_{x\to\infty} F(x,y)}_{= F_Y(y)} \geq F(x,y), \forall x, \forall y$$

$$\Rightarrow \underbrace{\lim F(x,y)}_{= F_X(x)} \geq F(x,y), \forall x, \forall y$$

$$\Longrightarrow G(x,y) \geq F(x,y), \forall x, \forall y$$

Milez

$X_1, X_2, \ldots$ n.e.p. diskretne

N sluč. sprem. tisto naravno št. n, za katerega je $X_n > X_1$

Ali ima N matematično upanje? $\rightarrow$ X celošt., $X > 0$

vaje: X celošt., $X > 0 \Longrightarrow E(X) = \sum_{j=1}^{\infty} P(X \geq j)$

$$E(|N|) = E(N) = \sum_{j=1}^{\infty} P(N \geq j) = \sum_{j=1}^{\infty} P(X_2 \leq X_1, X_3 \leq X_1, \ldots, X_{j-1} \leq X_1) =$$

$$\overset{X_i \text{ neodvisne}}{=} \sum_{j=1}^{\infty} P(X_2 \leq X_1) P(X_3 \leq X_1) \ldots P(X_{j-1} \leq X_1) \underset{\text{e.p.}}{=} \sum_{j=1}^{\infty} \underbrace{P(X_2 \leq X_1)}_{= q}{}^{j-2} < \infty$$

$q < 1$

geom. vrsta

$\Uparrow$ $P(X_2 < X_1) \neq 1$

$\Leftarrow$ $X_i \neq$ konst.

Peter

(X,Y) sluč. vektor

F porazd. fn. za (X,Y); F_X, F_Y marginalni por. fn. Tvorimo $G(x,y) = \max\{F_X + F_Y - 1, 0\}$. Izkaže se, da je G spet por. fn. nekega sluč. vektorja. Dokaži:

(a) G ima enaki marginalni por. fn. kot F

(b) $F(x,y) \leq G(x,y)$

(a) • $G_Y(y) = \lim_{x\to\infty} G(x,y) = \max\{\underbrace{\lim_{x\to\infty} F_X(x)}_{\to 1} + F_Y(y) - 1, 0\} =$ $x\to\infty$ ker F_X por. fn.

$= \max\{\underbrace{F_Y(y)}_{\geq 0}, 0\} = F_Y(y)$ ✓

• $G_X(x) = \lim_{y\to\infty} G(x,y) = \max\{\lim_{y\to\infty}(F_X(x) + \underbrace{F_Y(y)}_{\to 1} - 1), 0\} \overset{F_X \geq 0}{=} F_X(x)$ ✓ $y\to\infty$ ker F_Y por. fn.

(b) $F(x,y) = P(\{X<x\}\cdot\{Y<y\})$

$P(\overline{\{X<x\}\cdot\{Y<y\}}) \overset{\text{vemo}}{=} 1 - P(\{X<x\}\{Y<y\}) = \underline{1 - F(x,y)}$

$\|$

$P(\{X\geq x\}\cup\{Y\geq y\}) \overset{\text{vemo}}{=} \underbrace{P(\{X\geq x\})}_{1-F_X(x)} + \underbrace{P(\{Y\geq y\})}_{1-F_Y(y)} - P(\{X\geq x\}\cdot\{Y\geq y\})$

$\Rightarrow F(x,y) = \cancel{1} - \cancel{1} + F_X(x) - 1 + F_Y(y) + \underbrace{P(\{X\geq x\}\cdot\{Y\geq y\})}_{\geq 0} \geq$

$\geq F_X(x) + F_Y(y) - 1$

Ker vedno velja $F(x,y) \geq 0$, je $F(x,y) \geq G(x,y)$.

WTF

Ana

$$F(x,y)=\begin{cases} 0 \,;\ x\le 0 \text{ ali } y\le 0 \\ \frac{1}{2}(1-e^{-x}) \,;\ x>0 \text{ in } 0<y\le 1 \\ 1-e^{-x} \,;\ x>0,\ y>1 \end{cases}$$

F je skupna por. fu. sluč. vekt. (X,Y).

(a) Marginalni por. fu. $=?$

(b) Ali $\exists$ skupna gostota verjetnosti?

(a) • $F_X(x)=\lim\limits_{y\to\infty} F(x,y) = \begin{cases} 0 \,;\ x\le 0 \\ 1-e^{-x} \,;\ x>0 \end{cases}$ $\Rightarrow (y>1)$

• $F_Y(y)=\lim\limits_{x\to\infty} F(x,y) = \begin{cases} y\le 0 \,;\ 0 \\ 0<y\le 1 \,;\ \frac{1}{2} \\ y>1 \,;\ 1 \end{cases}$ $\Rightarrow x>0$

(b) x fiksen in $x>0$; $y=1 \Rightarrow F(x,1)=\frac{1}{2}(1-e^{-x})$ 2. predpis

$\lim\limits_{y\downarrow 1} F(x,y) = 1-e^{-x}$ $\times$

$\Rightarrow F$ ima skok $\Rightarrow F$ ni zv. $\Rightarrow F$ ni odvedljiva

$\Rightarrow F$ ni integral necesa $\Rightarrow$ $\nexists$ skupna gostota verj.

(oz.: $\frac{\partial^2 F(x,y)}{\partial x \partial y} = 0 \longrightarrow \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx = 0$ $*$ $\Rightarrow \nexists f$)

$\frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$ če $\exists$ gostota; $\int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx = F(x,y)$

$\boxed{\text{Mateja}}$ & Jaja

$F(x)$ por. f. neke sluč. sprem., $h>0$

$G(x)=\frac{1}{2h}\int_{x-h}^{x+h}F(t)\,dt$ je tudi por. f. neke sluč. sprem.

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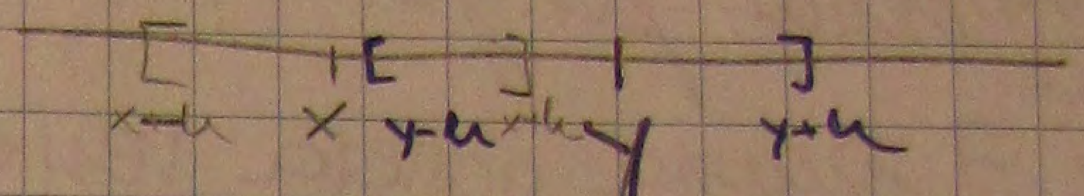
(1) • $x,y\in\mathbb{R}$; $x<y \Rightarrow G(x)\le G(y)$

$x-h<y-h$ in $x+h<y+h$



$\Rightarrow \int_{x-h}^{x+h}F(t)\,dt \underset{F\ nar.}{\le} \int_{y-h}^{y+h}F(t)\,dt \Rightarrow G(x)\le G(y)$ ✓

(2) • $\lim_{x\to-\infty}G(x)=0$ (lim $\exists$, ker G monotono pada ko $x\to-\infty$ (po (1)))

$G(x) \underset{F\ nar.}{\le} \frac{1}{2h}\int_{x-h}^{x+h}F(x+h)\,dt = \frac{F(x+h)}{2h}(x+h-x+h) = F(x+h) \xrightarrow{x\to-\infty} 0$

$\Rightarrow \lim_{x\to-\infty}G(x)\le 0$. Ker $G\ge 0$, je $\lim_{x\to-\infty}G(x)=0$ ✓

• $\lim_{x\to\infty}G(x)=1$ (G nar. ko $x\to\infty \Rightarrow$ lim $\exists$)

$G(x) \overset{F\le 1}{\le} \frac{1}{2h}\int_{x-h}^{x+h}1\,dt=\frac{2h}{2h}=1 \Rightarrow \lim_{x\to\infty}G(x)\le 1$

$G(x) \overset{F\ nar.}{\ge} \frac{1}{2h}\int_{x-h}^{x+h}F(x-h)\,dt=\frac{F(x-h)}{2h}2h=F(x-h) \xrightarrow{x\to+\infty} 1 \Rightarrow \lim_{x\to\infty}G(x)\ge 1$

$\Rightarrow \lim_{x\to+\infty}G(x)=1$ ✓

(3) G z leve zvezna $\forall x\in\mathbb{R} \Leftrightarrow G(x)=\lim_{y\uparrow x}G(y)$, $\forall x$

F z leve zv., $\int$ zv. $\Rightarrow$ G z leve zv.] bolje da to vidiš!

G monotona $\Rightarrow$ limita $\exists$ $\rightarrow$ dovolj izračunati na nekem podzaporedju.

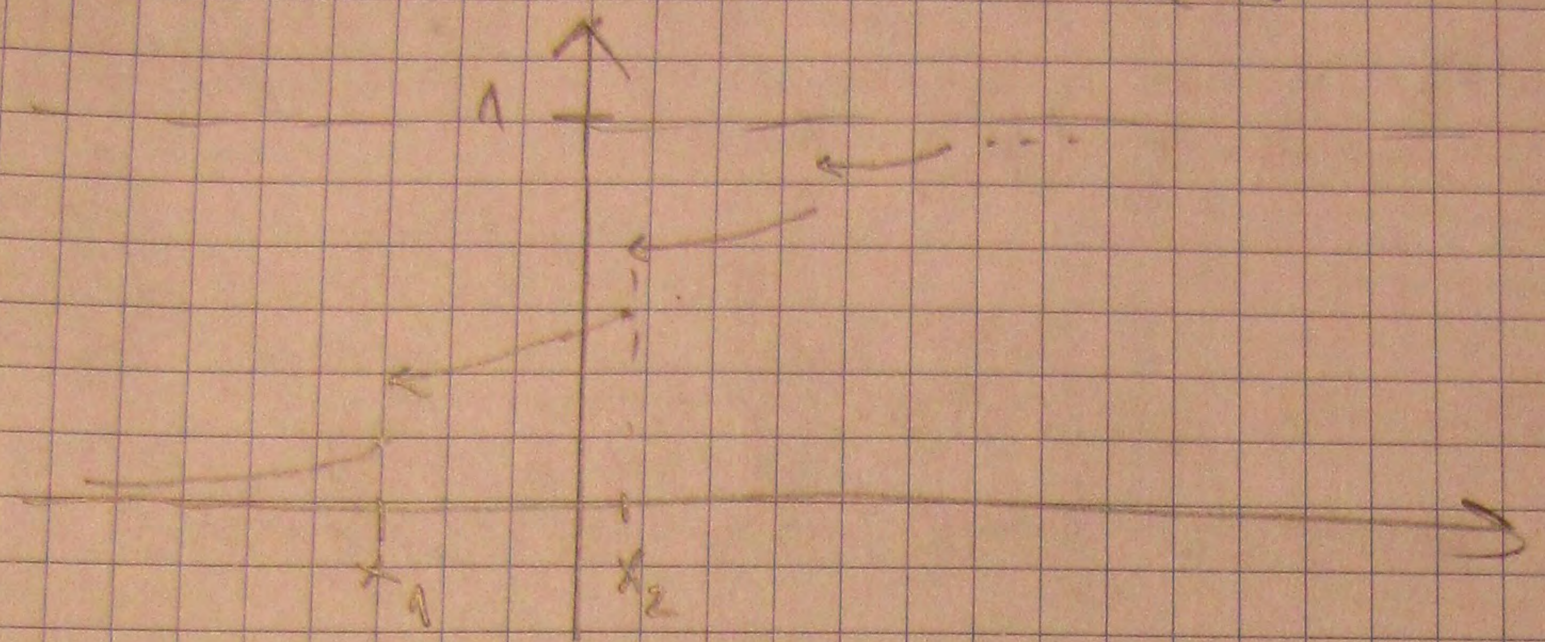
.....

M.PIRJIC → od zgoraj X

F por. fn. Dokaži, da $\exists$ diskretna spremen. X_1 z zakonom F_1 in zv. spremen. X_2 z zakonom F_2 in λ $\ni$: $F = \lambda F_1 + (1-\lambda) F_2$

$F(x) = P(\{X < x\})$ = zv. z leve zv.

IMA ŠTEVNO MNOGO SKOKOV



Naj bodo $\{x_i\}_{i=1}^{\infty} \subseteq \mathbb{R}$ točke, kjer ima F skok.

Za $i = 1, \dots, n$ naj bo λ_i velikost skoka v x_i

$$\Rightarrow \lambda_i = \lim_{x \downarrow x_i} F(x) - F(x_i)$$

Očitno je $\sum_{i=1}^{\infty} \lambda_i \leq 1$, ker $0 \leq F \leq 1$.

• Definiramo X_1 (diskretna): $X_1 \sim \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i=1}^{\infty}$ uni x_i od zgori!, kjer je

$$p_i := \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} \leq 1 \checkmark \qquad \cdot\ \lambda := \sum_{i=1}^{\infty} \lambda_i$$

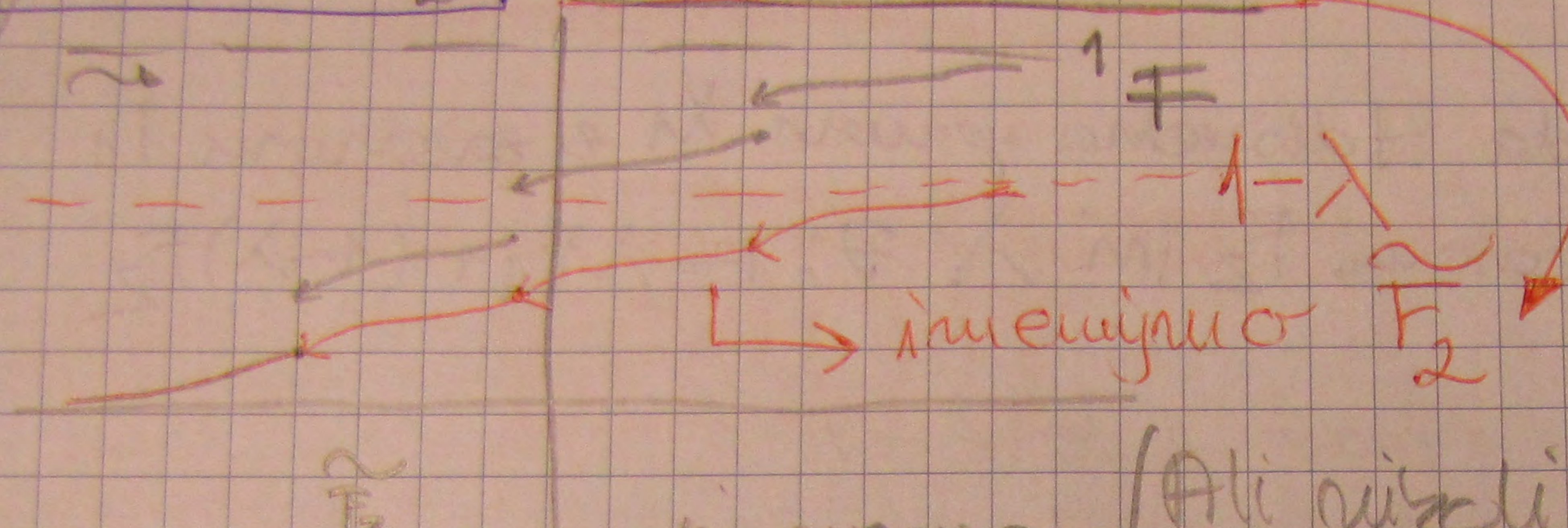
$$\left(\hookrightarrow \sum_{i=1}^{\infty} p_i = \frac{\sum_{i=1}^{\infty} \lambda_i}{\sum_{i=1}^{\infty} \lambda_i} = 1 \checkmark \right)$$

• $\Rightarrow F_1(x) = P(X_1 < x) = P\Big(\bigcap_{\{i \text{ tak, da } x_i < x\}} \{X_1 = x_i\}\Big) = \sum_{i=1}^{m(x)} p_i =$

↳ označimo $m(x)$ je max. indeks $\ni$: $x_{m(x)} < x$

$$= \frac{\sum_{i=1}^{m(x)} \lambda_i}{\lambda}$$

Definiramo F_2: skoke uničimo



$F_2 := \frac{\tilde{F}_2}{1-\lambda}$ je zvezna (Ali rišeš mi kolen? če F: ⇒ F_2: in to ni odvedljiva)

$$\lambda F_1(x) + (1-\lambda) F_2(x) = \sum_{i=1}^{n(x)} \lambda_i + \tilde{F}_2(x) \underset{\text{po konstrukciji}}{=} F(x)$$

?

X_2 "zgleda" zv. Potreben in zadosten pogoj je, da je P_{X_2} absolutno zvezen glede na Lebesgueovo mero.

Naj bo $B \subseteq \mathbb{R}$ Bor. mn. in $\overset{\text{leb. mera!}}{m}(B) = 0$

verjetno tole OK!

~~FA II.~~

~~I. Lokalno Konveksni Prostori~~

~~II.~~ $\boxed{\text{KARLA}}$

C

$d(X,Y) = E\left(\frac{|X-Y|}{1+|X-Y|}\right)$, X,Y sluč. sprem.

1) d metrika
2) Kaj je konv. zap. tle?
3) dosljeni pr. pdu

1) • $\frac{|X-Y|}{1+|X-Y|} \geq 0 \Rightarrow d(X,Y) \geq 0, \forall X,Y$

• $X=Y \to d(X,Y)=0$ ✓ $(E(0)=0)$

• $d(X,Y)=0 \Rightarrow P\left(\frac{|X-Y|}{1+|X-Y|}=0\right)=1$

$\Updownarrow$

$|X-Y|=0 \Leftrightarrow X-Y=0 \Leftrightarrow X=Y$ ✓

• $d(X,Y)=d(Y,X)$ ✓

• Δ neenak. $\qquad |x-y| \geq ||x|-|y||$

$$\frac{|X-Y|}{1+|X-Y|} - \frac{|X-Z|}{1+|X-Z|} - \frac{|Y-Z|}{1+|Y-Z|} =$$

$$= \frac{|X-Y|(1+|X-Z|+|Y-Z|+|X-Z||Y-Z|) - |X-Z|(1+|X-Y|+|Y-Z|+|X-Y||Y-Z|) +}{(1+|X-Y|)(1+|X-Z|)(1+|Y-Z|)}$$

$$\frac{-|Y-Z|(1+|X-Y|+|X-Z|+|X-Y||X-Z|)}{} =$$

$$= \frac{|X-Y| - |X-Z| - |Y-Z| - 2|X-Z||Y-Z| - |X-Y||X-Z||Y-Z|}{(1+|X-Y|)(1+|X-Z|)(1+|Y-Z|)} \leq$$

$$|X-Z+Z-Y| \leq |X-Z| + |Z-Y|$$

$$\leq -\frac{2|X-Z||Y-Z| + |X-Y||X-Z||Y-Z|}{(1+|X-Y|)(1+|X-Z|)(1+|Y-Z|)} \leq 0 \Rightarrow \frac{|X-Y|}{1+|X-Y|} \leq \frac{|X-Z|}{1+|X-Z|} + \frac{|Y-Z|}{1+|Y-Z|}$$

$$\to E\left(\frac{|X-Y|}{1+|X-Y|}\right) \underset{E \text{ monoton}}{\leq} E\left(\frac{|X-Z|}{1+|X-Z|} + \frac{|Y-Z|}{1+|Y-Z|}\right) \overset{E \text{ lin}}{=} d(X,Z) + d(Z,Y)$$ ✓

$\quad \parallel$

$d(X,Y)$

Akta

2) $X_n \xrightarrow{d} X \iff d(X_n, X) \to 0$

$\iff E\left(\frac{|X_n - X|}{1 + |X_n - X|}\right) \to 0$

$\iff \forall \varepsilon > 0 \ \exists n_0 \ni: \forall n \geq n_0:$

$E\left(\frac{|X_n - X|}{1 + |X_n - X|}\right) \leq \varepsilon$

(pon. konv. ki porojena iz vsake metrike (tega sicer ne vemo))

$\forall \delta$

$P(|X_n - X| \geq \delta) \leq \varepsilon$

Mogoče bi iz tega sledila konv. v verj.

JST

Konstruiraj take slučajne sprem. X, Y, Z, da je
$$E(E(X|Y)|Z) \neq E(E(X|Z)|Y)$$ s pozitivno verjetnostjo.

$X := Y$

I Vržemo 2 kocki: Y = manjše število
Z = večje število

sta neodvisni in ne moremo izraziti ene kot fu druge!

$$E(E(\overset{Y}{X}|Y)|Z) = E(Y|Z) \to \text{fu } Z\text{-ja}$$

$$\neq$$

$$E(E(X|Z)|Y) \to \text{fu } Y\text{-na}$$

II. $\Omega = 2^{\{1,2,3\}}$, $\mathcal{A}_1 = \{\{1,2\},\{3\},\emptyset,\Omega\}$, $\mathcal{A}_2 = \{\{1,3\},\{2\},\emptyset,\Omega\}$

($\mathcal{A}_1, \mathcal{A}_2$ sta σ-algebri) Definiramo

$$Y(\omega) = \begin{cases} 1 & ; \omega = 1,2 \\ 0 & ; \omega = 3 \end{cases} \quad \text{in} \quad Z(\omega) = \begin{cases} 1 & ; \omega = 2 \\ 0 & ; \omega = 1,3 \end{cases}$$

- $E(E(\overset{\shortparallel}{X}|Y)|Z=1) = E(Y|Z=1) = 1 \cdot P(Y=1|\underset{\substack{\shortparallel \\ \omega=2 \\ \Rightarrow Y=1}}{Z=1}) + 0 \cdot P(Y=0|Z=1) = \boxed{1}$

  $E(X|Y) = Y$

- $E(\overset{\overset{Y}{\shortparallel}}{X}|Z=1) \overset{\text{zgoraj}}{=} 1$
- $E(Y|Z=0) = 1 \cdot P(Y=1|\underset{\substack{\Downarrow \\ \omega = 1 \text{ ali } 3}}{Z=0}) = \frac{1}{2}$

- $E(E(X|Z)|Y=1) = 1 \cdot P(\underset{\substack{\shortparallel \\ 1/2}}{E(X|Z)=1}|\underset{\substack{\Downarrow \\ \omega = 1 \text{ ali } 2}}{Y=1}) + \frac{1}{2} P(\underset{\substack{\shortparallel \\ 1/2}}{E(X|Z)} = \frac{1}{2}|Y=1) = \frac{1}{2}+\frac{1}{4} = \boxed{\frac{3}{4}}$

  $\Rightarrow Z = 1$ z verj. $1/2$

  in $Z = 0$ z verj. $1/2$

- $E(E(X|Z)|Y=0) = 1 \cdot P(\underset{\substack{\shortparallel \\ 0}}{E(X|Z)=1}|\underset{\substack{\Downarrow \\ \omega=3 \\ \Rightarrow Z=0}}{Y=0}) + \frac{1}{2} P(\underset{\substack{\shortparallel \\ 1/2}}{E(X|Z)=\frac{1}{2}}|Y=0) = \frac{1}{4}$

- $E(E(X|Y)|Z=0) = E(Y|Z=0) = 1 \cdot P(Y=1|\underset{\substack{\Downarrow \\ \omega=1 \text{ ali } 3}}{Z=0}) = \frac{1}{2}$

Akta

A

$\mathcal{G}$ σ podalg. σ-alg $\mathcal{F}$

$X \geq 0$, $X \notin L^1 \Rightarrow E(|X|) = \infty$ $= E(X)$

$\exists!$ $\mathcal{G}$-merlj. Y $\ni$: $\int \chi_A X dP = \int \chi_A Y dP$, $\forall A$ iz $\mathcal{G}$

$E(\chi_A X)$ $\quad$ $E(\chi_A Y)$

Vemo: če $X \in L^1 \Rightarrow \exists!$ ..., da velja za vsaj $\mathcal{G}$-merlj. fn (ne le χ_A, $A \in \mathcal{G}$)

$\boxed{Y_n := E(\min(n, X) \mid \mathcal{G})}$

Y_n obstaja (ni je $\mathcal{G}$ slučajna sprem.), saj je $\min(n, X) \in L^1$

$\left(\int \min(n, X)\, dP \leq \int n\, dP = E(n) = n < \infty, \forall n\right)$

velja a

$\int \chi_A \min(n, X)\, dP = \int \chi_A Y_n\, dP$, $\forall A \in \mathcal{G}$, $\forall n$

Definiramo $\boxed{Y = \lim_{n\to\infty} Y_n}$, po točkah

$\min(n, X) \xrightarrow[\text{po točkah}]{n\to\infty} X$

v LMK nastopa konv. po točkah!

$\int \chi_A Y dP = \int \chi_A (\lim_{n\to\infty} Y_n) dP = \int \lim_{n\to\infty} (\chi_A Y_n) dP =$

$\min(n, X) \leq \min(n+1, X) \Longrightarrow Y_n \leq Y_{n+1}$, $\forall n \Rightarrow \chi_A Y_n \leq \chi_A Y_{n+1}$, $\forall n$

pogojno pričak. vrednost je monotona

$X \geq 0 \Rightarrow \min(n, X) \geq 0 \Longrightarrow Y_n \geq 0$, $\forall n \Rightarrow \chi_A Y_n \geq 0$, $\forall n$

LMK

$= \lim_{n\to\infty} \underbrace{\int \chi_A Y_n dP}_{\text{|| zgoraj}} \overset{\text{LMK}}{=} \int \lim_{n\to\infty} (\chi_A \min(n, X)) dP = \int \chi_A \lim_{n\to\infty} \min(n, X) dP =$

LMK ($\chi_A \min(n, X) \leq \chi_A \min(n+1, X)$ in $\chi_A \min(n, X) \geq 0$)

$\int \chi_A \min(n, X) dP$

$= \int \chi_A X dP$ ✓

KLARA

$X_n \Rightarrow X$, $Y_n \xRightarrow{?} c$ (zent). Ali $X_n Y_n \xRightarrow{?} X \cdot c$

Po premisleku: vzemi $X_n \geq 0$, $Y_n \geq 0$ in $c = 0$)

$\varphi_{X_n}(t) = E(e^{itX_n}) \to E(e^{itX}) = \varphi_X(t)$ po t

$\varphi_{Y_n}(t) = E(e^{itY_n}) \to E(e^{itc}) = e^{itc}$ po t

$\varphi_{X_n Y_n}(t) = E(e^{itX_nY_n}) \to E(e^{itX_n c}) \to E(e^{itXc})$

(?) Ali ok?

Luc

$X \in L^2$ ep in parome nekorelirane, $X_n \sim X$

$\frac{S_n}{n} \xrightarrow{L^2} a = E(X)$

$\|\frac{S_n}{n} - E(X)\|_2 = \|\frac{S_n}{n} - E(\frac{S_n}{n})\|_2 = \left(E\left(\left(\frac{S_n}{n} - E(\frac{S_n}{n})\right)^2\right)\right)^{1/2} = D(\frac{S_n}{n})^{1/2} =$

$E(\frac{S_n}{n}) = \frac{E(X_1)+\dots+E(X_n)}{n} \underset{ep}{=} \frac{nE(X)}{n} = E(X)$

$= \frac{1}{n} D(S_n)^{1/2} \underset{nekorelirane}{=} \frac{\sqrt{D(X_1)+\dots+D(X_n)}}{n} \underset{ep.}{=} \frac{\sqrt{nD(X)}}{n} = \frac{\sqrt{D(X)}}{\sqrt{n}} \xrightarrow{n\to\infty} 0$ ✓

Ursa

$X_n \Rightarrow X$, $X_n - Y_n \Rightarrow 0$. Ali velja $Y_n \Rightarrow X$?

$X_n \Rightarrow X \iff E(e^{itX_n}) \to E(e^{itX})$, $\forall t$

$X_n - Y_n \Rightarrow 0 \iff E(e^{it(X_n-Y_n)}) \to E(e^{it0}) = E(1) = 1$, $\forall t$

$\quad \| \; E(e^{itX_n} \cdot e^{-itY_n})$

Naj bosta X_n, Y_n neodvisni $\forall n$ /

$E(e^{itX_n} e^{-itY_n}) = E(e^{itX_n}) E(e^{-itY_n}) \Rightarrow E(e^{-itY_n}) \to \frac{1}{E(e^{itX})}$, $\forall t$

$\downarrow \quad\quad \downarrow$

$1 \quad\quad E(e^{itX})$

$\Rightarrow E(e^{itY_n}) \to \frac{1}{E(e^{-itX})}$

RSD: $\frac{1}{E(e^{-itX})} = E(e^{itX})$

Kar ustreza: $\frac{1}{\int f^{-1}} = \int f$ WTF

∃ Slutskyjev izrek:

$X_n \Rightarrow X$, $Y_n \Rightarrow c \Rightarrow X_n + Y_n \Rightarrow X + c$

Mi: $X_n \Rightarrow X$, $X_n - Y_n \Rightarrow 0$ $\xRightarrow{S.i.}$ $X_n + Y_n - X_n = Y_n \Rightarrow X + 0 = X$ ✓

$\Rightarrow Y_n - Y_n \Rightarrow 0$

MATEJA 3

X sluč. sprem., $X \geq 0$, X celošt., $P(X=0) = p \in (0,1)$.

$X_1, X_2, \dots$ n.e.p.; $X_i \sim X$, $\forall i$

$T = \min\{n \in \mathbb{N};\ X_n \neq 0\}$

Skupna por. sluč. vekt. $(T, X_T) = ?$

$\boxed{k \neq 0}$

$p_{n,k} = P(T=n, X_T = k) = P(T=n, X_n = k) = P(X_1 \neq 0, \dots, X_n = k \neq 0) =$ neodvisne (n.e.p.)

$= P(X_1 \neq 0) \dots P(X_{n-1} \neq 0) P(X_n = k \neq 0) \overset{\text{e.p.}}{=} (P(X=0))^{n-1} P(X=k \neq 0) =$

$= p^{n-1} P(X=k)$

$\boxed{k=0}$

$p_{n,0} = P(T=n, X_T = 0) = P(T=n, X_n = 0) = 0$

$$\Rightarrow \boxed{(T, X_T) \sim \begin{pmatrix} (n,k) \\ p_{n,k} \end{pmatrix}_{\substack{n=1,2,\dots \\ k=0,1,2,\dots}}, \text{ kjer } p_{n,k} = p^{n-1} P(X=k) \text{ za } k \neq 0 \text{ in } p_{n,0} = 0}$$

MATEJA 2

$X \sim I[0,1]$, $Y = 1 - X$. Določi skupno porazdelitev vektorja (X,Y) ter obe marginalni fn. Ali $\exists$ skupna gostota por?

$\tilde{F}_{X,Y}(x,y) = P(X < x, \underset{\substack{\text{II} \\ 1-y < X}}{1 - X < y}) = P(1-y < X < x) \underset{X \text{ zv. sluč. sprem.}}{=} F_X(x) - F_X(1-y)$

$$F_X(x) = \int_{-\infty}^{x} \chi_{[0,1]} dt = \begin{cases} x \leq 0; & 0 \\ 0 \leq x \leq 1; & \int_{-\infty}^{x} \chi_{[0,1]} dt = \int_0^x dt = x \\ x \geq 1; & \int_{-\infty}^{x} \chi_{[0,1]} dt = \int_0^1 dt = 1 \end{cases}$$

$$F_X(1-y) = \begin{cases} y \leq 0 \ (\Rightarrow 1-y \geq 1) & 1 \\ 0 \leq y \leq 1 \ (0 \leq 1-y \leq 1) & 1-y \\ y \geq 1 \ (1-y \leq 0) & 0 \end{cases}$$

$$\Rightarrow \boxed{\tilde{F}_{X,Y}(x,y) = \begin{cases} x, y \leq 0; & -1 \\ x \leq 0, 0 \leq y \leq 1; & y-1 \\ x \leq 0, y \geq 1 & 0 \\ 0 \leq x \leq 1, y \leq 0 & x-1 \\ 0 \leq x \leq 1, 0 \leq y \leq 1 & x+y-1 \\ 0 \leq x \leq 1, y \geq 1 & x \\ x \geq 1, y \leq 0 & 0 \\ x \geq 1, 0 \leq y \leq 1 & y \\ x \geq 1, y \geq 1 & 1 \end{cases}}$$

X zv✓

- $\tilde{F}_X(x) = \lim_{y\to\infty} \tilde{F}_{X,Y}(x,y) = F_X(x) - \overbrace{\lim_{y\to\infty} F_X(\underbrace{1-y}_{\to -\infty})}^{0} = F_X(x)$

- $\tilde{F}_Y(y) = \lim_{x\to\infty} \tilde{F}_{X,Y}(x,y) = \underbrace{\lim_{x\to\infty} F_X(x)}_{1} - F_X(1-y) = 1 - F_X(1-y) = F_Y(y)$

$F_Y(y) = P(Y<y) = P(1-X<y) = P(1-y<X) = 1 - P(X \le 1-y) \overset{zv}{=} 1 - F_X(1-y)$

oziroma $\tilde{F}_Y(y) = 1 - F_X(1-y) = \begin{cases} y \le 0; & 1-1=0 \\ 0\le y \le 1; & 1-(1-y)=y \\ y \ge 1; & 1-0=1 \end{cases} = F_X(y)$

- $\underbrace{\dfrac{\partial^2 \tilde{F}_{X,Y}(x,y)}{\partial x \partial y}}_{f(x,y) \text{ če } \exists} = 0 \Rightarrow \tilde{F}_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(t,s)\,ds\,dt = 0$ ✗

$\Longrightarrow$ ne obstaja zvezna gostota porazdelitve

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① $\mathcal{G}$ podalgebra, σ-algebra $\mathcal{F}$, X nenegativna slučajna spremenljivka. $\forall \varepsilon > 0$ velja: $\varepsilon\, P(X \geq \varepsilon \mid \mathcal{G}) \leq E(X \mid \mathcal{G})$

NEENAKOST MARKOVA

$$\varepsilon \cdot 1_{(X \geq \varepsilon)} \leq X \quad / \cdot E \text{ pogojno } \mathcal{G}$$

$$E(\varepsilon \cdot 1_{(X\geq\varepsilon)} \mid \mathcal{G}) \leq E(X \mid \mathcal{G})$$

$$\varepsilon \cdot E(1_{(X\geq\varepsilon)} \mid \mathcal{G}) \leq E(X\mid\mathcal{G})$$

$$\varepsilon\, \overset{(*)}{=} P(X \geq \varepsilon \mid \mathcal{G}) \leq E(X \mid \mathcal{G})$$

(*) definicija $P(A \mid \mathcal{G}) = E(\chi_A \mid \mathcal{G})$

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② Konstruiraj take tri slučajne spremenljivke X, Y in Z, da $E(E(X\mid Y)\mid Z)$ s pozitivno verjetnostjo ni enako $E(E(X\mid Z)\mid Y)$.

1) Izbereš lahko 0 ali 1 enako verjetno
2) še enkrat izbereš 0 ali 1 enako verjetno

X ... št. enic
Y ... najmanjše izbrano št.
Z ... največje izbrano št.

1) • $E(X \mid Y=0) = \frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 0 = \frac{2}{3}$

možnosti		X
0	0	0
0	1	1
1	0	1

da bo $X=1$ je $P(X=1) = \frac{2}{3}$
$X=0$ je $P(X=0) = \frac{1}{3}$

$E(X \mid Y=1) = 2$ — če $Y=1$ najmanjše potem smo izbrali 2 enici

$E(X\mid Y) = \frac{2}{3} + \frac{4}{3}\cdot Y$ (ko $Y=0$ je $E(X\mid Y=0) = \frac{2}{3}$ in $E(X\mid Y=1)=2$)

• $E(\frac{2}{3} + \frac{4}{3}Y \mid Z) = \frac{2}{3} + \frac{4}{3}E(Y\mid Z) = \frac{2}{3} + \frac{4}{3}\cdot\frac{Z}{3} = \underline{\underline{\frac{2}{3} + \frac{Z4}{9}}}$

$$\left.\begin{aligned} E(Y\mid Z=0) &= 0 \\ E(Y\mid Z=1) &= \tfrac{2}{3}\cdot 0 + \tfrac{1}{3}\cdot 1 = \tfrac{1}{3}\end{aligned}\right\} E(Y\mid Z) = \frac{Z}{3}$$

2) • $E(X\mid Z=0) = \frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 0 = \frac{2}{3}$
$E(X\mid Z=1) = 2\cdot\frac{1}{3} + \frac{2}{3}\cdot 1 = \frac{4}{3}$

		za X		
1	1	$x=2$	$x=2$	$\frac{1}{3}$
1	0		$x=1$	$\frac{2}{3}$
0	1			

$E(X\mid Z) = \frac{2}{3} + \frac{2}{3}Z$

• $E(\frac{2}{3} + \frac{2}{3}Z \mid Y) = \frac{2}{3} + \frac{2}{3}E(Z\mid Y) = \underline{\frac{2}{3} + \frac{2}{3}\cdot\frac{2}{3} + \frac{2}{3}Y}$

		za Z
0	0	0
0	1	1
1	0	1
1	1	

$$\left.\begin{aligned} E(Z\mid Y=0) &= 0\cdot\tfrac{1}{3} + 1\cdot\tfrac{2}{3} = \tfrac{2}{3} \\ E(Z\mid Y=1) &= 1\end{aligned}\right\} E(Z\mid Y) = \frac{2}{3} + Y$$

ni enako

③ Naj bo $\mathcal{G}$ σ-podalgebra σ-algebre $\mathcal{F}$, X pa naj bo nenegativna sluč. sprem., ki ni v L^1. Pokaži, da $\exists$ ena $\mathcal{G}$-merljiva sluč. sprem. Y, za katero velja $\int \chi_A X \, dP = \int \chi_A Y \, dP$ za vse A iz $\mathcal{G}$.

Vzemimo $X_n = \begin{cases} X, & |X| < n \\ 0, & \text{sicer} \end{cases}$

$Y_n = E(X_n | \mathcal{G})$ (Y_n naraščajoča)

Vemo, da velja: $\int \chi_A X_n \, dP = \int \chi_A Y_n \, dP$

Ker Y_n naraščajoča, recimo, da je $Y_n \to Y$.

(LEBEGOV INTEGRAL)

$$\lim_{n\to\infty} \int \chi_A X_n \, dP = \lim_{n\to\infty} \int \chi_A Y_n \, dP$$

$$\int \chi_A \lim_{n\to\infty} X_n \, dP = \int \chi_A \lim_{n\to\infty} Y_n \, dP$$

$$\int \chi_A X \, dP = \int \chi_A Y \, dP$$

Samo še dokažemo, da $\exists! \, Y$.

Recimo, da $\exists$ dva.

$$\int \chi_A X \, dP = \int \chi_A Y_1 \, dP$$

$$\int \chi_A X \, dP = \int \chi_A Y_2 \, dP$$

$$\int \chi_A Y_1 \, dP = \int \chi_A Y_2 \, dP$$

$$\int \chi_A (Y_1 - Y_2) \, dP = 0$$

- Naj bo A mn. kjer je $Y_1 > Y_2$ $\Rightarrow P(A) = 0$
- Naj bo A mn. kjer je $Y_1 < Y_2$ $\Rightarrow P(A) = 0$

$\Rightarrow Y_1 = Y_2$ ($P(Y_1 = Y_2) = 1$) skoraj gotovo

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(7) Naj bo $\mathcal{G}$ -podalgebra $\mathcal{G}$-algebre $\mathcal{F}$, X in Y pa dve slučajni spremenljivki iz L^2. Pokaži, da velja

$$(E(XY|\mathcal{G}))^2 \leq E(X^2|\mathcal{G})\,E(Y^2|\mathcal{G})$$

$$\langle X,Y\rangle = E(X\cdot Y\,|\,\mathcal{G}) \qquad (\text{CAUCHY-SHWARZ})$$

↓ pokaži, da je skalarni produkt

- $\langle X,X\rangle \geq 0$
- $\langle X,Y\rangle = 0$
- $\langle X,Y\rangle = \langle Y,X\rangle$
- $\langle \alpha X,Y\rangle = \alpha\langle X,Y\rangle$
- $\langle X,X\rangle = 0 \Leftrightarrow X = 0$ s.g.
- $\langle X+Y,Z\rangle = \langle X,Z\rangle + \langle Y,Z\rangle$

(8) Naj za slučajni spremenljivki X in Y iz L^1 velja, da je skoraj gotovo $E(X|Y) = X$ in $E(Y|X) = Y$. Pokaži, da sta spremenljivki skoraj gotovo enaki

X, Y sta skoraj gotovo enaki: $P(X=Y)=1$

Dve sprem. imata isto mate. upanje, če sta skoraj gotovo enaki.

$E(X|Y) = X$ s.g. $\Rightarrow E(E(X|Y)) = E(X)$

$E(Y|X) = Y$ s.g. $\Rightarrow \underbrace{E(E(Y|X))}_{E(Y)} = E(Y)$

X poljubna sluč. sprem.

$Y = 2X$

$E(X|Y) = E(X|2X) = \frac{2X}{2} = X$

$E(Y|X) = E(2X|X) = 2X = Y$

(8') $E(X|Y) = Y$ in $E(Y|X) = X$ s.g.

Recimo da sta $X, Y \in L^2$

$E((X-Y)^2) = 0$ (X in Y s.g. enaka)

$E(X^2 - 2XY + Y^2) = E(X^2) - 2E(XY) + E(Y^2) = 0$

$E(X \cdot Y) = E(E(X \cdot Y|Y)) = E(Y\, E(X|Y)) = E(Y \cdot Y) = E(Y^2)$

$E(XY) = E(E(XY|X)) = E(X\, \underbrace{E(Y|X)}_{X}) = E(X^2)$

$E(|X-Y|) = 0$ za $X, Y \in L^1$

$E((X-Y) \cdot 1_{(X>Y)} + (X-Y) 1_{(X<Y)}) = 0$

⑥ Naj zaporedje porazdelitvenih funkcij $F_1, F_2, \ldots \Rightarrow F$, kjer je F neka zvezna porazdelitvena funkcija. Pokaži, da tedaj zaporedje konvergira proti F tudi v normi prostora L^∞.

$F_1, F_2, \ldots \to F$ zv. porazd. funk.

$$F_m \xrightarrow{\|\cdot\|_\infty} F$$

$$\|F_m - F\|_\infty \to 0$$

$(\not\Rightarrow)$

$$X_1, X_2, \ldots \xrightarrow{d} X$$

$$X_i \Rightarrow X \iff F_i \to F \iff \phi_{X_i} \to \phi_X \iff E(f(X_i)) \to E(f(X))$$

$\forall$ zv. om. f

$$E(|f(X_i) - f(X)|) = 0$$

L^∞ ?

Izberi $f(x) := \ldots$

$$F_n(x) = P(X_n < x)$$

$$\lim_{n\to\infty} \sup_x |P(X_n < x) - P(X < x)| = 0$$

$\Rightarrow$ še razmisli kaj vzameš za funkcijo f.

(še ni konec naloge)

(1) Naj bodo $X_1, X_2, \ldots$ neodvisne, enako porazdeljene diskretne sprem. Naj bo $N = \min\{n;\ X_n > X_1\}$ sluč. sprem. Ugotovi ali ima N matematično upanje.

$$P(N > n) = \frac{1}{n}$$

$$E(N) = \sum_{n=1}^{\infty} n P(N=n) = \sum_{n=1}^{\infty} P(N>n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$\Rightarrow$ N nima mat. up.

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(2) $X_1, X_2, \ldots$ nekorelirane iz L^2 z $E(X_1), E(X_2), \ldots = a$

$$\frac{S_n}{n} \xrightarrow{\|\cdot\|} a \quad (v\ L^2)$$

Isto kot če pokažeš, da je

$$E\left(\left(\frac{S_n}{n} - a\right)^2\right) = 0 \qquad \text{(dokončaj)}$$

— — — -

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(3) Naloga: $X_n \Rightarrow X$, $Y_n \Rightarrow c$ $\Rightarrow$ $X_n Y_n \Rightarrow c \cdot X$

$c = \text{konst}$

je SLUTSKY-JEVA TRDITEV

(pomagaj si z dokazom)