

KINEMATIKA TOGEGA TELESA

- Gibanje materialnih točk $P_i, i=1, \dots, N$ je topo, če se razdalje med točkami ohranjajo
 $|P_i(t) - P_j(t)| = C_{ij} \quad \forall t, \text{ za } \forall \text{ par } i, j$
- Sistem mat točk je tog, če je edini njegov način gibanja togo gibanje.
- Togo telo je materialno telo katerega edini način gibanja je togo gibanje.

$N=3$ $\cdot P_1$ $\cdot P_3$
 $\cdot P_2$

9 koordinat
 Topogibanje $\Rightarrow 3$ vezi

$\Downarrow$
 gibanje ima 6 neodvisnih parametrov

SPLOŠNO: za $\forall N$ ima gibanje 6 neodv. parametrov

$\cdot P_3$

3 za translacijo + 3 za rotacijo

$\cdot P_1$

$\cdot P_4$

$\cdot P_2$

$\cdot P_3'$

$\cdot P_4'$

$\cdot P_1'$

$\cdot P_2'$

žal. razdalje
 + orientacija

matanko določa P_4

IZREK Najbo gibanje sistema točk togo. Potem za $\forall$ koord. sist.

γ in poljubno točko P_0 obstajata funkciji $t \rightarrow P_0'(t)$

in $t \rightarrow Q(t) \in SO(3)$ tako, da je

a) $P_0'(t=0) = P_0, \quad Q(t=0) = I$

b) $P(t) = P_0'(t) + Q(t) \cdot (P(t=0) - P_0)$ za $\forall P$ iz sistema

DOKAZ

$\cdot P$ $t=0$

A

C

B

P_0

podaj v času t

$\cdot P'$

$\cdot C'$

$\cdot A'$

$\cdot B'$

$\cdot P_0'$

P_0, A, B, C miso kopečmarini

$$\vec{P_0 P} = \alpha \vec{P_0 A} + \beta \vec{P_0 B} + \gamma \vec{P_0 C} \quad / \quad \vec{P_0 A}, \vec{P_0 B}, \vec{P_0 C}$$

$$\vec{P_0' P'} = \alpha' \vec{P_0' A'} + \beta' \vec{P_0' B'} + \gamma' \vec{P_0' C'}$$

$$\underline{b} = \begin{cases} \vec{P_0 P} \cdot \vec{P_0 A} = \alpha \vec{P_0 A} \cdot \vec{P_0 A} + \beta \vec{P_0 B} \cdot \vec{P_0 A} + \gamma \vec{P_0 C} \cdot \vec{P_0 A} \\ \vec{P_0 P} \cdot \vec{P_0 B} = \dots \\ \vec{P_0 P} \cdot \vec{P_0 C} = \dots \end{cases}$$

$$\underline{\alpha} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \quad \underline{A} = \begin{bmatrix} \vec{P_0 A} \cdot \vec{P_0 A} \\ \vdots \\ \vec{P_0 C} \cdot \vec{P_0 C} \end{bmatrix} \Rightarrow \underline{b} = \underline{A} \cdot \underline{\alpha}$$

$$\underline{\alpha}' = \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix}, \quad \underline{A}' = \dots \Rightarrow \underline{b}' = \underline{A}' \cdot \underline{\alpha}'$$

$$\underline{b}' = \begin{bmatrix} \vec{P_0' P'} \cdot \vec{P_0' A'} \\ \vdots \end{bmatrix} = \underline{b} \quad \text{ker je gibanje togo je } \underline{b}' = \underline{b} \text{ in } \underline{A}' = \underline{A} \\ \Rightarrow \underline{\alpha}' = \underline{\alpha}$$

$$\vec{P_0 P} = P - P_0 \xrightarrow{\text{gibanje}} P' - P_0 \\ \text{je linearna mešikava}$$

$|\vec{P_0 P}| = |\vec{P_0' P'}|$ je mešikava tudi ortogonalna

$$\Rightarrow P' - P_0' = Q(P - P_0) \Rightarrow P' = P_0' + Q(P - P_0)$$

$$P(t) = \underbrace{P_0'(t)}_{P_0(t=0)} + Q \underbrace{(P(t=0) - P_0)}_{P_0(t=0)}$$

$$t=0; \Rightarrow Q(t=0) = I \\ P_0'(t=0) = P_0$$

gibanje točno $\Rightarrow \det Q$ zvezna

$$\det(Q(t=0)) = \det I = 1$$

$$\Rightarrow \det Q(t) = 1$$

$$\Rightarrow Q \in SO(3) \text{ (rotacija)}$$

$$P(t) = P_0'(t) + Q(t)(P(t=0) - P_0)$$

Togo gibanje lahko opišemo z relat. gibanjem ŽKS

$$\psi(P, t), \quad P = P(t=0) \text{ glede na AKS}$$

$$\psi'(P', t), \quad P' = P(t)$$

RKS se giblje hkrati s tagimi sistemom (telesom) zato mu

pravimo telesno koord. sistem (TKS)

Točke v TKS mirujejo.

V TKS je $\dot{\mathbf{P}} = \vec{v}_{rel} = \vec{0}$

sistem mat. točk $\rightarrow$ (telo)

$$P_1, \dots, P_N, m_1, \dots, m_N$$

$$m = \sum_{i=1}^N m_i$$

$$\mathbf{P}_* = \mathbf{P}_0 + \frac{1}{m} \sum_{i=1}^N m_i (\mathbf{P}_i - \mathbf{P}_0)$$

$$\vec{L} = \vec{L}(0) = \sum_{i=1}^N (\mathbf{P}_i - \mathbf{O}) \times m_i \dot{\mathbf{P}}_i$$

$B(t)$

$$m = m(B) = \int_B \rho dV$$

$$dm = \rho dV$$

dm sredi P

$$\mathbf{P}_* = \mathbf{P}_0 + \frac{1}{m} \int_B (\mathbf{P} - \mathbf{P}_0) dm(P)$$

$$\vec{L} = \vec{L}(0) = \int_B (\mathbf{P} - \mathbf{O}) \times \dot{\mathbf{P}} dm$$

$$\mathbf{P}_*' = \mathbf{P}_0' + \frac{1}{m} \int_B (\mathbf{P}' - \mathbf{P}_0') dm(P)$$

$$B' = B(t), B = B(t=0)$$

$$\vec{a}, \vec{a}' = Q \vec{a}$$

$$\mathbf{P}_* - \mathbf{P}_0 = \frac{1}{m} \int_B (\mathbf{P} - \mathbf{P}_0) dm(P), \quad \mathbf{P}_*' - \mathbf{P}_0' = \frac{1}{m} \int_{B'} (\mathbf{P}' - \mathbf{P}_0') dm(P')$$

$$\text{Ali je } (\mathbf{P}_*' - \mathbf{P}_0') = Q(\mathbf{P}_* - \mathbf{P}_0)?$$

$$\rightarrow \mathbf{P}' = \mathbf{P}_0' + Q(\mathbf{P} - \mathbf{P}_0)$$

$$\mathbf{P}_*' - \mathbf{P}_0' = \frac{1}{m} \int_{B'} (\mathbf{P}' - \mathbf{P}_0') dm(P') = \frac{1}{m} \int_{B'} Q(\mathbf{P} - \mathbf{P}_0) dm(P) =$$

$$= \frac{1}{m} \int_B Q(\mathbf{P} - \mathbf{P}_0) \left| \frac{\partial \mathbf{P}'}{\partial \mathbf{P}} \right| dm(P) = Q(\mathbf{P}_* - \mathbf{P}_0) \quad ; \quad Q = Q(t) \text{ krajeno neod.}$$

$\det Q = 1$

$dm(P') = dm(P)$ ker je masa naka za vse opazovalce

$$\boxed{\mathbf{P}_*' = \mathbf{P}_0' + Q(\mathbf{P}_* - \mathbf{P}_0)}$$

$$\frac{d(\mathbf{P}_*')}{dt} = \dot{\mathbf{P}}_*' = \frac{1}{m} \sum m_i \dot{\mathbf{P}}_i' \stackrel{?}{=} \frac{1}{m} \int_{B'} \dot{\mathbf{P}}' dm(P') = \dot{\mathbf{P}}_*'$$

$$\mathbf{P}_*' = \mathbf{P}_0' + \frac{1}{m} \int_B (\mathbf{P}' - \mathbf{P}_0') dm$$

$$\frac{d\mathbf{P}_*'}{dt} = \dot{\mathbf{P}}_0' + \frac{1}{m} \frac{d}{dt} \int_{B'=B(t)} (\mathbf{P}' - \mathbf{P}_0') dm$$

$$B(t=0) \mapsto B(t)$$

$$P \mapsto P'$$

$$\frac{d}{dt} \int_{B(t=0)} (\mathbf{P}' - \mathbf{P}_0') \left| \frac{\partial \mathbf{P}'}{\partial \mathbf{P}} \right| dm \stackrel{Q=1}{=} \frac{d}{dt} \int_{B(t=0)} (\mathbf{P}' - \mathbf{P}_0') dm - \int_{B(t=0)} (\dot{\mathbf{P}}' - \dot{\mathbf{P}}_0') dm =$$

$$= \int_{B'} (\dot{\mathbf{P}}' - \dot{\mathbf{P}}_0') dm$$

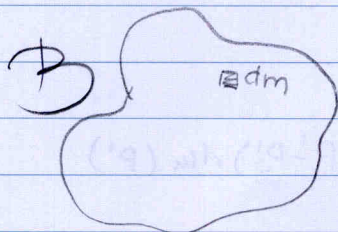
$$\begin{aligned} & \frac{d}{dt} \int_{B(t=0)} (\mathbf{P}' - \mathbf{P}_0') dm = \frac{d}{dt} \int_{B(t=0)} (\mathbf{P}' - \mathbf{P}_0') dm - \int_{B(t=0)} (\dot{\mathbf{P}}' - \dot{\mathbf{P}}_0') dm = \\ & \quad \left| \frac{\partial \mathbf{P}'}{\partial \mathbf{P}} = Q \right. \quad \mathbf{P}' = \mathbf{P}_0' + Q(\mathbf{P} - \mathbf{P}_0) \end{aligned}$$

$$\frac{d\mathbf{P}_*'}{dt} = \cancel{\dot{\mathbf{P}}_0'} + \frac{1}{2} \int_{B'} (\dot{\mathbf{P}}_i' - \cancel{\dot{\mathbf{P}}_0'}) dm = \frac{1}{m} \int_{B'} \dot{\mathbf{P}}_i' dm.$$

Trditev Velja

$$\boxed{\begin{aligned} \frac{d\mathbf{P}_*'}{dt} &= \frac{1}{m} \int_{B'} \dot{\mathbf{P}}_i' dm \\ \frac{d^2\mathbf{P}_*'}{dt^2} &= \frac{1}{m} \int_{B'} \ddot{\mathbf{P}}_i' dm. \end{aligned}}$$

Enačba gibanja masnega sred.: $m \ddot{\mathbf{P}}_*' = \vec{\mathbf{F}}' \leftarrow$
 $m_i \ddot{\mathbf{P}}_i' = \vec{\mathbf{F}}_i' + \sum_j \vec{\mathbf{F}}_{ji}'$



$$\ddot{\mathbf{P}} dm = \underbrace{d\vec{\mathbf{F}}(\mathbf{P})}_{\vec{\mathbf{f}}(\mathbf{P}) dm} + \int_B \vec{\mathbf{f}}(\mathbf{P}, \hat{\mathbf{P}}) dm(\hat{\mathbf{P}})$$

$\vec{\mathbf{f}}(\mathbf{P}) \dots$ gostota zun. sil.
 $\vec{\mathbf{f}}(\mathbf{P}, \hat{\mathbf{P}}) \dots$ gostota notr. sil.
 $\vec{\mathbf{f}}(\mathbf{P}, \hat{\mathbf{P}}) = -\vec{\mathbf{f}}(\hat{\mathbf{P}}, \mathbf{P})$

V AKS (pišemo zanadi enostavnosti linea črtice)

$$\int_B \ddot{\mathbf{P}} dm = \underbrace{\int_B \vec{\mathbf{f}}(\mathbf{P}) dm}_{=\vec{\mathbf{F}}'} + \underbrace{\int_B \int_B \vec{\mathbf{f}}(\mathbf{P}, \hat{\mathbf{P}}) dm(\hat{\mathbf{P}}) dm(\mathbf{P})}_{=0 \text{ ker } \vec{\mathbf{f}}(\mathbf{P}, \hat{\mathbf{P}}) = -\vec{\mathbf{f}}(\hat{\mathbf{P}}, \mathbf{P})}$$

$$\boxed{m \ddot{\mathbf{P}}_*' = \vec{\mathbf{F}}}$$

$\vec{\mathbf{F}}$ volumenska sila

$$\vec{\mathcal{L}}(\mathbf{P}_*) = \int_B \underbrace{(\mathbf{P}_i' - \mathbf{P}_0')}_{\vec{\xi}_i'} \times (\mathbf{P}_i' - \mathbf{P}_0')^* dm = \int_B \vec{\xi}_i' \times \dot{\vec{\xi}}_i' dm$$

$$\mathbf{P}_i' = \underbrace{\mathbf{P}_0'}_{\vec{\xi}_0'} + \underbrace{Q(\mathbf{P} - \mathbf{P}_0)}_{\vec{\xi}}$$

$$\vec{\xi}_i' = Q \vec{\xi}, \quad \dot{\vec{\xi}}_i' = \dot{Q} \vec{\xi} + Q \underbrace{\dot{\vec{\xi}}_0'}_0$$

$$\begin{aligned} \dot{\vec{\xi}}_i' &= \dot{Q} \vec{\xi} = Q Q^T \dot{Q} \vec{\xi} = Q(\vec{\omega} \times \vec{\xi}) \\ \dot{\vec{\xi}}_i' &= \vec{\omega}' \times \vec{\xi}_i' \end{aligned}$$

ker se RKS
giblje hkrati
s telesom

$$\vec{L}'(P_0') = \int_{B'} \vec{\xi}' \times (\vec{\omega}' \times \vec{\xi}') d\mu(P')$$

$$\vec{\xi}' \times (\vec{\omega}' \times \vec{\xi}') = |\vec{\xi}'|^2 \vec{\omega}' - \underbrace{(\vec{\omega}' \cdot \vec{\xi}') \vec{\xi}'}_{\substack{\text{identični} \\ \text{tenzor} \\ \text{tenzorski} \\ \text{produkt}}} = \quad \left(\vec{a} \otimes \vec{b} \right) \vec{c} = (\vec{b} \cdot \vec{c}) \vec{a}$$

$$= (|\vec{\xi}'|^2 \underline{I} - \vec{\xi}' \otimes \vec{\xi}') \vec{\omega}'$$

$$\vec{L}'(P_0') = \left[\int_{B'} (|\vec{\xi}'|^2 \underline{I} - \vec{\xi}' \otimes \vec{\xi}') d\mu(P') \right] \vec{\omega}'$$

Urt. kol.
↓ gibanja okrog P_0'

$$\vec{L}'(P_0') = \underline{J}' \vec{\omega}'$$

↑ je funkcija položaja

$\underline{J}'$ vztrajnostni
momentni tenzor

$\underline{J}' = \underline{J}'(P_0')$ je odvisen
o položaja

telesni vztrajnostni
tenzor
 $\underline{J}(P_0)$

def. $\underline{L}(P_0) := Q^T \vec{L}'(P_0')$

$$\vec{L}'(P_0') = \underline{J}'(P_0') \vec{\omega}' = \underline{J}'(P_0') Q \vec{\omega}$$

$\parallel$
 $Q \underline{L}(P_0)$

$$\Rightarrow \underline{L}(P_0) = Q^T \underline{J}'(P_0') Q \vec{\omega}$$

$$\underline{L}(P_0) = \underline{J}(P_0) \vec{\omega}$$

$$\underline{J}(P_0) = \int_{B'} \left[\underbrace{Q^T |\vec{\xi}'|^2 Q}_{\parallel \vec{\xi}' \parallel^2 \underline{I}} - Q^T (\vec{\xi}' \otimes \vec{\xi}') Q \right] d\mu(P')$$

$$= (Q^T \vec{\xi}') \otimes (Q^T \vec{\xi}') \Rightarrow$$

$|\vec{\xi}'| = |\vec{\xi}|$

TRDITEV $\underline{A}(\vec{a} \otimes \vec{b}) = (\underline{A} \vec{a}) \otimes \vec{b}$
 $(\vec{a} \otimes \vec{b}) \circ \underline{A} = \vec{a} \otimes (\underline{A}^T \vec{b})$

DOKAZ

$$\bullet \underline{A}(\vec{a} \otimes \vec{b}) \cdot \vec{c} = \underline{A}((\vec{a} \otimes \vec{b}) \cdot \vec{c}) = \underline{A}((\vec{b} \cdot \vec{c}) \vec{a}) = (\vec{b} \cdot \vec{c}) \underline{A} \vec{a} =$$

$$= ((\underline{A} \vec{a}) \otimes \vec{b}) \cdot \vec{c} \quad \checkmark$$

$$\bullet ((\vec{a} \otimes \vec{b}) \circ \underline{A}) \cdot \vec{c} = (\vec{a} \otimes \vec{b}) (\underline{A} \vec{c}) = (\vec{b} \cdot \underline{A} \vec{c}) \vec{a} = (\vec{c} \cdot \underline{A}^T \vec{b}) \vec{a} = (\vec{a} \otimes \underline{A}^T \vec{b}) \cdot \vec{c}$$

$$\Rightarrow \underline{\underline{J}}(P_0) = \int_{B'} (|\vec{\xi}|^2 \underline{\underline{I}} - (\vec{\xi} \otimes \vec{\xi})) dm(P) = \int_B (|\vec{\xi}|^2 \underline{\underline{I}} - \vec{\xi} \otimes \vec{\xi}) dm$$

telesni vztr. tenzor

TRDITEV: $\underline{\underline{J}}$ je simetričen in pozitivno semidefiniten. Če B ni tanka palica, je pozitivno definiten.

$$\underline{\underline{J}} = \underline{\underline{J}}^T, \quad \underline{\underline{J}} \vec{a} \cdot \vec{a} \geq 0 \quad ; \quad \underline{\underline{J}} \vec{a} \cdot \vec{a} > 0, \quad \vec{a} \neq \vec{0}$$

DOKAZ tenz. produkt: $(\vec{a} \otimes \vec{b})^T = \vec{b} \otimes \vec{a}$

$$(\vec{a} \otimes \vec{b})^T \vec{u} \cdot \vec{v} = \vec{u} \cdot (\vec{a} \otimes \vec{b}) \vec{v} = \vec{u} \cdot (\vec{b} \cdot \vec{v}) \vec{a} = (\vec{u} \cdot \vec{a}) (\vec{b} \cdot \vec{v})$$

$$(\vec{b} \otimes \vec{a}) \vec{u} \cdot \vec{v} = \vec{b} \cdot (\vec{a} \cdot \vec{u}) \vec{v} = (\vec{a} \cdot \vec{u}) (\vec{b} \cdot \vec{v}) \quad //$$

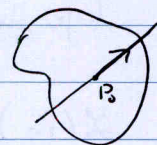
$$\underline{\underline{J}}^T = \int_B (|\vec{\xi}|^2 \underline{\underline{I}} - \vec{\xi} \otimes \vec{\xi})^T dm = \underline{\underline{J}} \quad \checkmark$$

$$\underline{\underline{J}} \vec{a} \cdot \vec{a} = \int_B (|\vec{\xi}|^2 |\vec{a}|^2 - (\vec{\xi} \cdot \vec{a})^2) dm \geq 0 \quad \forall \vec{a} \quad \checkmark$$

$\vec{\xi} \cdot \vec{a} \leq |\vec{\xi}| |\vec{a}|$

$$\underline{\underline{J}} \vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{\xi} \parallel \vec{a} \text{ skoraj povsod}$$

$\Rightarrow B$ tanka palica $\checkmark$



Definicija Lastnim vrednostim vztr. tenzorja pravimo glavne vrednosti, lastnim vektorjem pa glavne osi.
"glavni momenti"

V telesnem KS katere koord. se ujemajo z gl. osmi nprada $\underline{\underline{J}}$ diag. matrika z glavnimi momenti na diagonali.

Komponentni zapis vztraj. tenzorja:

$$J_{ij} = \vec{e}_i \cdot \underline{\underline{J}} \vec{e}_j = \int_B \vec{e}_i (|\vec{\xi}|^2 \underline{\underline{I}} - \vec{\xi} \otimes \vec{\xi}) \vec{e}_j dm = \int_B |\vec{\xi}|^2 \delta_{ij} -$$

o če izj, || M drugače

$$\vec{e}_i (\vec{\xi} \otimes \vec{\xi}) \vec{e}_j = (\vec{e}_i \cdot \vec{\xi}) (\vec{e}_j \cdot \vec{\xi})$$

$$J_{11} = \int_B (\xi_2^2 + \xi_3^2) dm \quad ; \quad J_{22} = \int_B (\xi_1^2 + \xi_3^2) dm \quad ; \quad J_{33} = \int_B (\xi_1^2 + \xi_2^2) dm$$

$$J_{12} = - \int \xi_1 \xi_2 dm = J_{21}$$

$$J_{13} = - \int \xi_1 \xi_3 dm = J_{31}$$

$$J_{23} = - \int \xi_2 \xi_3 dm = J_{32}$$

$$\underline{J_e = \vec{e} \cdot J \vec{e}} \quad \text{vztrajnostni moment} \\ |\vec{e}| = 1 \quad \text{okrog osi } \vec{e}.$$

Glavne osrednosti = vztr. momenti
okrog glavnih osi.

Primer

homogen valj

dm

$\rho = \text{konst.}$

$$J_{11} = \int_B (y^2 + z^2) \rho dV$$

cilindrične koord: $x = r \cos \vartheta$
 $y = r \sin \vartheta$
 $z = z$

$$J_{11} = \int_0^{2\pi} \int_0^a \int_{-h/2}^{h/2} (r^2 \sin^2 \vartheta + z^2) \rho r dr dz d\vartheta = \rho \left(\underbrace{h \int_0^a r^3 dr}_{\frac{1}{4} a^4} \int_0^{2\pi} \sin^2 \vartheta d\vartheta + \underbrace{2\pi \int_0^a r dr}_{\frac{1}{2} a^2} \int_{-h/2}^{h/2} z^2 dz \right) =$$

$$= \left(\frac{1}{4} a^2 h + 2 \frac{1}{2} a^2 \cdot \frac{2}{3} \frac{h^3}{8} \right) \rho \pi = \rho \pi a^2 h \left(\frac{a^2}{4} + \frac{h^2}{12} \right) ; m = \rho V = \rho \pi a^2 h$$

$$\underline{J_{11} = m \left(\frac{a^2}{4} + \frac{h^2}{12} \right)}$$

$J_{22} = J_{11}$ (simetrično na y osi kot po x osi)

$$J_{33} = \int_B (x^2 + y^2) \rho dV = \rho \int_0^{2\pi} d\vartheta \int_{-h/2}^{h/2} dz \int_0^a r r dr = 2\pi h \frac{1}{4} a^2 \rho = (\pi a^2 h \rho) \frac{1}{2} a^2 = \underline{\underline{\frac{1}{2} m a^2}}$$

vztr. moment okrog z-osi

$$J_{12} = - \int_B xy \rho dV = -\rho \int_0^{2\pi} d\vartheta \int_{-h/2}^{h/2} dz \int_0^a r dr \cdot r^2 \frac{1}{2} \sin 2\vartheta = 0$$

od 0-2π → 0

$$J_{13} = - \int_B xz \rho dV = 0 \quad \int_0^{2\pi} \cos \vartheta d\vartheta = 0 \quad ; \quad \underline{J_{23} = 0}$$

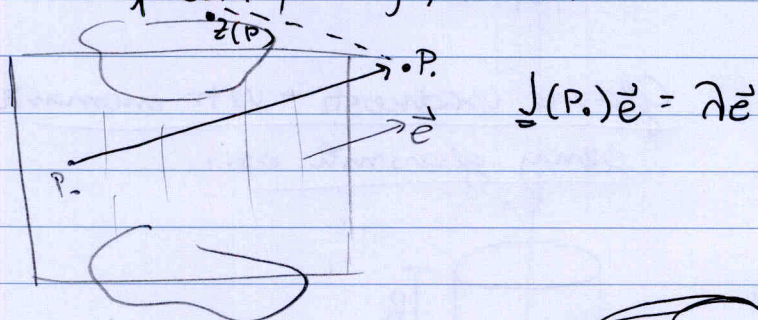
$$J_{ij} = - \int_B \xi_i \xi_j dm$$

$$\underline{D = - \int_B (\vec{\xi} \otimes \vec{\xi}) dm} \quad \text{deviator}$$

če vzamemo pravilno koord. sistem
mat. je diagonalna. (glede na simetri. telesa)

Dij deviacijski momenti

TRDITEV Če ima telo ravnamo zrcalne simetrije, je normala na to ravnamo glavna smer (= os) vztr. tenzorja s polom na tej ravlini.



DOKAZ

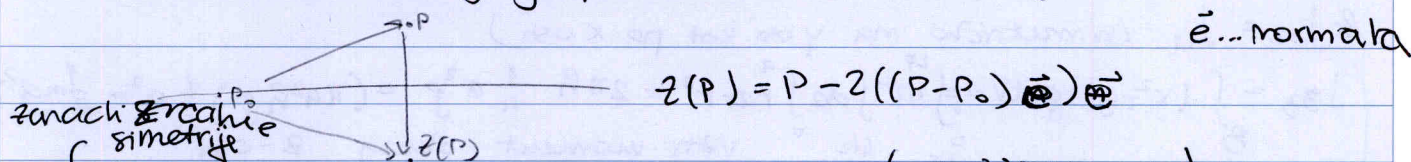
$$J(P_0) = \int_B |P - P_0|^2 I \, d\mu = \int_B (P - P_0) \otimes (P - P_0) \, d\mu \quad \Leftarrow$$

$$J(P_0) \vec{e} = \left(\int_B |P - P_0|^2 \, d\mu \right) \vec{e} + D \vec{e}$$

$D \vec{e} \parallel \vec{e}$ $\Leftrightarrow D \vec{e} \cdot \vec{f} = 0 \quad \forall \quad \vec{f} \perp \vec{e}$:

$$D \vec{e} \cdot \vec{f} = - \int_B \underbrace{(P - P_0) \otimes (P - P_0) \vec{e}}_{(P - P_0) \vec{e} (P - P_0)} \cdot \vec{f} \, d\mu = - \int_B ((P - P_0) \vec{e}) ((P - P_0) \vec{f}) \, d\mu = *$$

$P \mapsto Z(P)$ zrcaljenje preko ravnine simetrije



zrcaljenje preko ravnine simetrije

$$* = - \int_B ((Z(P) - P_0) \vec{e}) ((Z(P) - P_0) \vec{f}) \, d\mu = * \left(\left| \frac{\partial Z(P)}{\partial P} \right| = 1 \right)$$

$$\begin{aligned} (Z(P) - P_0) \vec{e} &= (P - P_0) \vec{e} - 2(P - P_0) \vec{e} (\vec{e} \cdot \vec{e}) = -(P - P_0) \vec{e} \\ (Z(P) - P_0) \vec{f} &= (P - P_0) \vec{f} \end{aligned}$$

$$** = + \int_B ((P - P_0) \vec{e}) (P - P_0) \vec{f} \, d\mu = - D \vec{e} \cdot \vec{f} = D \vec{e} \cdot \vec{f} \Leftrightarrow D \vec{e} \vec{f} = 0.$$

Trditve Presek dveh ravnin zrc. simetrije je glavna os, vztrajnostnega tenzorja.

2 ravnine lastni podprosti (ravnine)

DOKAZ

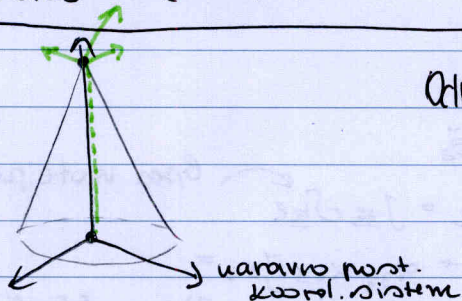
$$J \vec{e}_1 = \lambda_1 \vec{e}_1 \quad J \vec{e}_2 = \lambda_2 \vec{e}_2 \quad \text{Im } \vec{e}_1 \perp \vec{e}_2$$

$\vec{e}_3$ v smeri preseka ravnin $\vec{e}_3 \perp \vec{e}_1, \vec{e}_3 \perp \vec{e}_2$

$$J \vec{e}_3 = \lambda_3 \vec{e}_3$$

$$\vec{e}_i \cdot \int \vec{e}_3 = 0, \quad i=1,2$$

$$\int \vec{e}_i \vec{e}_3 = (\int \vec{e}_i) \vec{e}_3 = 0. \quad \checkmark$$



Odlisnost vtr. tenzorja od pola:

$$\underline{J}(P_0) = \int (|P-P_0|^2 \underline{I} - (P-P_0) \otimes (P-P_0)) d\mu =$$

B

$$(P-P_0) = (P-P_*) + (P_*-P_0)$$

$$= \int (|P-P_*|^2 |P_*-P_0|^2 + 2(P-P_*)(P_*-P_0)) d\mu =$$

$$3 \int \underline{J}(P_*) \cdot \int (P-P_*) \otimes (P_*-P_0) + (P-P_*) \otimes (P_*-P_0) + (P_*-P_0) \otimes (P-P_*) +$$

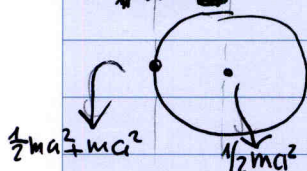
$$(P_*-P_0) \otimes (P_*-P_0) d\mu = \underline{J}(P_*) + |P_*-P_0|^2 m \underline{I} - m(P_*-P_0) \otimes (P_*-P_0) = \underline{J}(P_0)$$

Steinerjev izrek

$$\int (P-P_*) d\mu \stackrel{?}{=} \vec{0}$$

$$P_* = P_0 + \frac{1}{m} \int (P-P_0) d\mu \quad \text{če izberemo } P_0 \text{ je } P_*$$

ZA Krog



$$\vec{e}_3 = \int_* \vec{e}_3 = 1/2 ma^2$$

$$\vec{e}_3 \cdot \int (P_0) \vec{e}_3 = 1/2 ma^2 + ma^2$$

$$P_0 - P_* = a \vec{e}_1$$

$$(P_* - P_0) = -a \vec{e}_1, \quad (P_* - P_0) = a \vec{e}_1$$

$$|P_* - P_0|^2 = a^2$$

Odlisnost od orientacije koord. sistema:

$$\vec{e}_i; \quad J_{ij} = \vec{e}_i \cdot \int \vec{e}_j$$

Kozi koord. sistem = bazisni vektorji: $\vec{e}_i = \sum_j \alpha_{ij} \vec{e}_j = \alpha_{ij} \vec{e}_j$

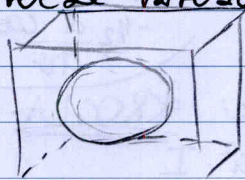
$$J_{ij} = \vec{e}_i \cdot \int \vec{e}_j = \alpha_{ik} \vec{e}_k \cdot \int \alpha_{je} \vec{e}_e = \alpha_{je} \int \alpha_{ik} \vec{e}_k \cdot \vec{e}_e$$

$$J_{ij} = \alpha_{ik} \alpha_{je} \int \vec{e}_k \cdot \vec{e}_e$$

$$\alpha_{ik} \cdot \alpha_{jk} = \delta_{ij} \quad (1 \text{ ali } 0)$$

sumacijska konvencija

12 kocke vrezana krogla

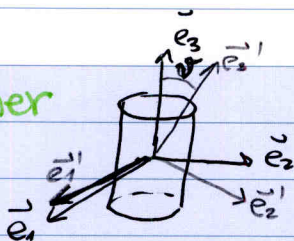


$$J = ?$$

$$J_{\text{kocka}} - J_{\text{krogla}}$$

$$J_{\text{stožec na kocki}} = \text{vsota } J_{\text{st}} + J_{\text{kocka}}$$

Primer



$\vec{e}_1', \vec{e}_2', \vec{e}_3'$ dobimo rotacijo okrog osi $\vec{e}_3$ za kot ϑ .

$$\vec{e}_1' = \vec{e}_1$$

$$\vec{e}_2' = -\sin \vartheta \vec{e}_2 + \cos \vartheta \vec{e}_3$$

$$\vec{e}_3' = \sin \vartheta \vec{e}_2 + \cos \vartheta \vec{e}_3$$

Izračunaj J_{33}' .

$$J_{ke} = J_{\underline{k}} \delta_{\underline{k}\underline{e}} \quad \text{brez vsote po } k$$

$$J_{33}' = \alpha_{3k} \alpha_{3e} J_{ke} = \alpha_{31} \alpha_{31} J_{11} + \alpha_{32} \alpha_{32} J_{22} + \alpha_{33} \alpha_{33} J_{33} =$$

$$= \sin^2 \vartheta J_{22} + \cos^2 \vartheta J_{33} = \sin^2 \vartheta (1/4 m a^2 + 1/12 m b^2) + \cos^2 \vartheta \cdot 1/2 m a^2$$

$$\Rightarrow J_{11}' = J_{11}$$

$$\begin{cases} \vec{e}_i' = \alpha_{ij} \vec{e}_j; & \vec{e}_3' = \alpha_{32} \vec{e}_2 + \alpha_{33} \vec{e}_3 \\ \alpha_{31} = 0; & \alpha_{32} = \sin \vartheta; \alpha_{33} = \cos \vartheta \end{cases}$$

Primer Vztr. moment elipsoida (2 Routhovo metodo)

$$J_{11} = \int_B ((\xi_2)^2 + (\xi_3)^2) dm$$

$$\begin{aligned} \int_B (\xi_1)^2 dm &= \int_B a^2 x^2 (abc) \rho dx dy dz = \\ &= (abc) a^2 \rho \underbrace{\int_B x^2 dx dy dz}_{K_x} \cdot \frac{1}{|B|} |B| = (abc) a^2 \rho |B| K_x \end{aligned}$$

$$\begin{cases} \xi_1 = ax \\ \xi_2 = by \\ \xi_3 = cz \\ \hat{B} \mapsto B \\ (x, y, z) \mapsto (\xi_1, \xi_2, \xi_3) \end{cases}$$

$$\Rightarrow \int_B (\xi_1)^2 dm = m a^2 K_x$$

$$\int_B (\xi_2)^2 dm = m b^2 K_y$$

$$\int_B (\xi_3)^2 dm = m c^2 K_z$$

B elipsoid $\Rightarrow B$ krogla $K_x = K_y = K_z$

$$K_z = \frac{1}{4\pi} \int z^2 dx dy dz = \frac{3}{4\pi}$$

$$= \frac{3}{4\pi} \int_0^1 r^2 dr \int_0^{2\pi} d\varphi \int_0^\pi \sin \vartheta (r^2 \cos^2 \vartheta) = \frac{3}{4\pi} 2\pi \cdot \frac{1}{3} \int_0^\pi \sin \vartheta \cos^2 \vartheta d\vartheta$$

$$= \frac{3 \cdot 2\pi}{4\pi \cdot 5 \cdot 3} (-1)(-2) = \frac{1}{5}$$

$$J_{11} = \frac{m}{5} (b^2 + c^2)$$

$$\Rightarrow J_{22} = \frac{m}{5} (a^2 + c^2)$$

$$J_{33} = \frac{m}{5} (a^2 + b^2)$$

POSEBEN PRIMER: KROGLA

$$a=b=c: J = \frac{2}{5} m a^2 \cdot I$$

sferični tenzor ($d \cdot I$)

Centrosimetr. telo - telo, ki ima sferični tenzor (krogla, kodača)

KINETIČNA ENERGIJA TOGEGA TELESJA

$$T = \frac{1}{2} \sum_{i=1}^N m_i |\dot{\mathbf{P}}_i'|^2 \longrightarrow \frac{1}{2} \int_{B'} |\dot{\mathbf{P}}'|^2 dm(\mathbf{P}')$$

$$\mathbf{P}' = \mathbf{P}_0' + \mathbf{Q} (\mathbf{P} - \mathbf{P}_0)$$

$$\dot{\mathbf{P}}' = \dot{\mathbf{P}}_0' + \dot{\boldsymbol{\omega}}' \times (\mathbf{P}' - \mathbf{P}_0) \quad (\mathbf{P}_0 = \mathbf{P}_*)$$

$$\dot{\mathbf{P}}' = \dot{\mathbf{P}}_*' + \dot{\boldsymbol{\omega}}' \times (\mathbf{P}' - \mathbf{P}_*')$$

$$|\dot{\mathbf{P}}'|^2 = |\dot{\mathbf{P}}_*'|^2 + \underbrace{2 \dot{\mathbf{P}}_*' \cdot (\dot{\boldsymbol{\omega}}' \times (\mathbf{P}' - \mathbf{P}_*'))}_{\text{integral tega} = 0} + |\dot{\boldsymbol{\omega}}' \times (\mathbf{P}' - \mathbf{P}_*')|^2$$

$$T = \frac{1}{2} m |\dot{\mathbf{P}}_*'|^2 + \frac{1}{2} \int_{B'} |\dot{\boldsymbol{\omega}}' \times (\mathbf{P}' - \mathbf{P}_*')|^2 dm$$

$$\begin{aligned} (\dot{\boldsymbol{\omega}}' \times (\mathbf{P}' - \mathbf{P}_*')) \cdot (\dot{\boldsymbol{\omega}}' \times (\mathbf{P}' - \mathbf{P}_*')) &= |\dot{\boldsymbol{\omega}}'|^2 |\mathbf{P}' - \mathbf{P}_*'|^2 - (\dot{\boldsymbol{\omega}}' \cdot (\mathbf{P}' - \mathbf{P}_*'))^2 \\ &= \dot{\boldsymbol{\omega}}' \cdot (|\mathbf{P}' - \mathbf{P}_*'|^2 \mathbf{I}) \dot{\boldsymbol{\omega}}' - \dot{\boldsymbol{\omega}}' \cdot ((\mathbf{P}' - \mathbf{P}_*') \otimes (\mathbf{P}' - \mathbf{P}_*')) \dot{\boldsymbol{\omega}}'. \end{aligned}$$

$$T = \frac{1}{2} m |\dot{\mathbf{P}}_*'|^2 + \frac{1}{2} \dot{\boldsymbol{\omega}}' \cdot \mathbf{J}' (\mathbf{P}_*') \dot{\boldsymbol{\omega}}'$$

kin. energ. marnega
središča

rotacijska kin. energija

$$\mathbf{J}' = \mathbf{Q} \mathbf{J} \mathbf{Q}^T \quad \dot{\boldsymbol{\omega}}' = \mathbf{Q} \dot{\boldsymbol{\omega}}$$

$$\frac{1}{2} \dot{\boldsymbol{\omega}}' \mathbf{J}' \dot{\boldsymbol{\omega}}' = \frac{1}{2} (\mathbf{Q} \dot{\boldsymbol{\omega}}) (\mathbf{Q} \mathbf{J} \mathbf{Q}^T) (\mathbf{Q} \dot{\boldsymbol{\omega}}) = \frac{1}{2} \dot{\boldsymbol{\omega}} \mathbf{J} \dot{\boldsymbol{\omega}}$$

$$\dot{\mathbf{v}}_*' = \dot{\mathbf{P}}_*' \quad \dot{\mathbf{v}}_* = \mathbf{Q}^T \dot{\mathbf{v}}_*'$$

$$\Rightarrow T = \frac{1}{2} m |\dot{\mathbf{v}}_*|^2 + \frac{1}{2} \dot{\boldsymbol{\omega}} \mathbf{J} \dot{\boldsymbol{\omega}}$$

Izrek o energiji

$$\frac{dT}{dt} = \sum_{i=1}^N \underbrace{\vec{\mathbf{F}}_i \cdot \dot{\mathbf{P}}_i}_{\text{moč zun. sil}} + \frac{1}{2} \sum_{i,j} \underbrace{\vec{\mathbf{F}}_{ij} (\dot{\mathbf{P}}_i - \dot{\mathbf{P}}_j)}_{\text{moč notr. sil}}$$

Trditev: Pri togem gibanju je moč centralnih sil enaka nič. notranjih

Dokaz: $\sum_{i,j} \vec{F}_{ij}' (\dot{\vec{P}}_i' - \dot{\vec{P}}_j') = 0$

$$\vec{P}_i' = \vec{P}_0'(t) + Q(t) (\vec{P}_i - \vec{P}_0)$$

$$\dot{\vec{P}}_i' = \dot{\vec{P}}_0' + (\dot{\vec{P}}_i - \dot{\vec{P}}_0)$$

$$\dot{\vec{P}}_i' - \dot{\vec{P}}_j' = \vec{\omega}' \times (\vec{P}_i' - \vec{P}_j') ; \quad \vec{F}_{ji}' \parallel (\vec{P}_i' - \vec{P}_j').$$

za togo telo: $\frac{dT}{dt} = \int_{B'} \vec{f}' \cdot \dot{\vec{P}}_i' dm = \vec{f}$ gostota zun. sil

$$= \int_{B'} \vec{f} \cdot (\dot{\vec{P}}_0' + \vec{\omega}' \times (\vec{P}_i' - \vec{P}_0')) dm = \int_{B'} \vec{f}' \cdot \dot{\vec{P}}_i' dm + \int_{B'} \vec{f}' \cdot (\vec{\omega}' \times (\vec{P}_i' - \vec{P}_0')) dm$$

$$\Rightarrow \frac{dT}{dt} = \underbrace{\left(\int_{B'} \vec{f} dm \right)}_{=\vec{F}'} \dot{\vec{P}}_0' + \underbrace{\vec{\omega}' \int_{B'} (\vec{P}_i' - \vec{P}_0') \times \vec{f} dm}_{\vec{N}'(\vec{P}_0')}$$

rezultanta
zun. sil

$$\vec{N}'(\vec{P}_0')$$

$$\sum (\vec{P}_i - \vec{P}_0) F_i = \vec{N}(\vec{P}_0).$$

(navor)

Trditev izrek o energiji:

$$\frac{dT}{dt} = \vec{F} \cdot \dot{\vec{P}}_0' + \vec{\omega}' \cdot \vec{N}'(\vec{P}_0')$$

translatorska
moč. zun. sil

rotacijska moč
zun. sil

Izrek o vrtilni količini

$$\frac{d\vec{L}'(\vec{O}')}{dt} = \vec{N}'(\vec{O}')$$

$$\frac{d\vec{L}'(\vec{P}_0')}{dt} = \vec{N}'(\vec{P}_0') ; \quad \vec{P}_0' = \begin{cases} \vec{P}_0' \\ \vec{P}_* \end{cases} \text{ mirujoča točka (telo se vrtili okoli } \vec{P}_0')$$

$$\vec{L}'(\vec{P}_0') = \int (\vec{P}_i' - \vec{P}_0') \times \vec{p}_i' dm = Q \int \vec{r} \times \vec{p} dm$$

$$\frac{d\vec{L}'}{dt} = Q \int \dot{\vec{r}} \times \vec{p} dm + Q \int \vec{r} \times \dot{\vec{p}} dm = Q \int \vec{r} \times \vec{p} dm + Q \int \vec{r} \times \vec{p} dm$$

$$= Q \int \vec{r} \times \vec{p} dm = Q (\vec{\omega} \times \int \vec{r} dm + \int \vec{r} \times \vec{p} dm)$$

$$\vec{N}'(\vec{P}_0') = \int_{B'} \vec{f} \times (\vec{P}_i' - \vec{P}_0') dm = \int_{B'} \vec{f} \times (\vec{P}_i - \vec{P}_0) dm = \int_{B'} \vec{f} \times (\vec{P}_i - \vec{P}_0) dm$$

$$= \int_{B'} (Q \vec{f} \times Q (\vec{P}_i - \vec{P}_0)) dm(\vec{P}') = Q \int_{B'} (\vec{f} \times (\vec{P}_i - \vec{P}_0)) dm = Q \int_{B'} (\vec{f} \times (\vec{P}_i - \vec{P}_0)) dm$$

$$\vec{N}(\vec{P}_0)$$

$$\Rightarrow \vec{N}'(\vec{P}_0') = Q (\vec{N}(\vec{P}_0))$$

$$\frac{d\vec{L}'(P_0')}{dt} = \vec{N}'(P_0') \Rightarrow Q(\vec{\omega} \times \underline{J}(P_0) \vec{\omega} + \underline{J}(P_0) \dot{\vec{\omega}}) = Q \vec{N}(P_0)$$

$$\boxed{\vec{\omega} \times \underline{J}(P_0) \vec{\omega} + \underline{J}(P_0) \dot{\vec{\omega}} = \vec{N}(P_0)}$$

Eulerjeve dinamične enačbe

Dinamika tega telesa je dana z enačbo gibanja mas.redišča $m \ddot{\vec{r}}_*' = \vec{F}'$ in z Eulerjevimi din. enačbami:

Eulerjeve din. enačbe v lastni bazi vstr. tenzorja:

$$\underline{J} = \sum_{i=1}^3 J_i \vec{e}_i \otimes \vec{e}_i \quad J_{ij} = \vec{e}_i \cdot \underline{J} \vec{e}_j$$

$$\vec{\omega} = \sum_{j=1}^3 \omega_j \vec{e}_j$$

$$\dot{\vec{\omega}} = \sum_{j=1}^3 \dot{\omega}_j \vec{e}_j$$

$$\vec{\omega} \times \underline{J} \vec{\omega} = \left(\sum_{i=1}^3 \omega_i \vec{e}_i \right) \times \left(\sum_{j=1}^3 J_j \vec{e}_j \otimes \vec{e}_j \right) \sum_{k=1}^3 \omega_k \vec{e}_k =$$

$$\sum_{j,k=1}^3 J_j \omega_k \underbrace{\vec{e}_j (\vec{e}_j \otimes \vec{e}_k)}_{\substack{\text{vsota po } i \\ \text{po } k}} = \sum_{k=1}^3 J_k \omega_k \vec{e}_k$$

$$= \sum_{k=1}^3 \omega_i \omega_k J_k (\vec{e}_i \times \vec{e}_k) \quad (\text{vsota tudi po } i) =$$

$$\boxed{\vec{e}_i \times \vec{e}_k = \epsilon_{ikj} \vec{e}_j}$$

$$\text{permutacija} \quad \epsilon_{ikj} = \begin{cases} 0 & , \text{ ikj ni permutacija } 123 \\ 1 & , \text{ ikj je soda perm} \\ -1 & , \text{ ikj je liha} \end{cases}$$

$$= \sum_{k=1}^3 \omega_i \omega_k J_k \epsilon_{ikj} \vec{e}_j = \left(\sum_{k=1}^3 \omega_i \omega_k J_k \epsilon_{ik1} \right) \vec{e}_1 + \left(\sum_k \omega_i \omega_k J_k \epsilon_{ik2} \right) \vec{e}_2 +$$

$$+ \left(\sum_k \omega_i \omega_k J_k \epsilon_{ik3} \right) \vec{e}_3 =$$

$$= (\omega_2 \omega_3 J_3 \underset{1}{\overset{11}{\epsilon_{231}}} + \omega_3 \omega_2 J_2 \underset{-1}{\overset{11}{\epsilon_{321}}}) \vec{e}_1 + (\omega_1 \omega_3 J_3 \underset{-1}{\overset{11}{\epsilon_{132}}} + \omega_3 \omega_1 J_1 \underset{1}{\overset{11}{\epsilon_{312}}}) \vec{e}_2 +$$

$$+ (\omega_1 \omega_2 J_2 \underset{1}{\overset{11}{\epsilon_{123}}} + \omega_2 \omega_1 J_1 \underset{-1}{\overset{11}{\epsilon_{213}}}) \vec{e}_3$$

po komponentah:

$$\begin{cases} J_1 \dot{\omega}_1 - \omega_2 \omega_3 (J_2 - J_3) = N_1 \\ J_2 \dot{\omega}_2 - \omega_3 \omega_1 (J_3 - J_1) = N_2 \\ J_3 \dot{\omega}_3 - \omega_1 \omega_2 (J_1 - J_2) = N_3 \end{cases}$$

oznanka: $\vec{\omega}$

Eulerjeve enačbe
v lastnem
koord. sistemu

Eulerjeve din. enačbe za rot. okrog stabilne osi

$\vec{\omega}' = \vec{\omega} = \dot{\varphi} \vec{e}_3$ — maviloma
v tej bazi J mi diagonalizirani

$$\begin{aligned} \vec{e}_1 &\perp \vec{e}_3 \\ \vec{e}_2 &\perp \vec{e}_3 \end{aligned}$$

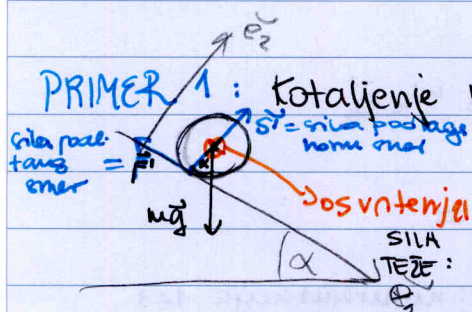
$$\vec{\omega} \times J \vec{\omega} = N \vec{e}_3$$

$$\vec{\omega} = \dot{\varphi} \vec{e}_3 \quad ; \quad J \vec{\omega} = \dot{\varphi} (J_{13} \vec{e}_1 + J_{23} \vec{e}_2 + J_{33} \vec{e}_3)$$

$$\vec{\omega} \times J \vec{\omega} = \dot{\varphi}^2 \vec{e}_3 (J_{13} \vec{e}_1 + J_{23} \vec{e}_2 + J_{33} \vec{e}_3) = \dot{\varphi}^2 (J_{13} \vec{e}_2 - J_{23} \vec{e}_1)$$

$$\begin{aligned} \bullet \quad \ddot{\varphi} J_{13} - \dot{\varphi}^2 J_{23} &= N_1 \\ \bullet \quad \ddot{\varphi} J_{23} + \dot{\varphi}^2 J_{13} &= N_2 \\ \bullet \quad \ddot{\varphi} J_{33} &= N_3 \end{aligned}$$

neznanaka φ



zun. sila = sila teže, sila podlage

$$\begin{aligned} \vec{F}' &= \int_{B'} \vec{f} dm = \int_{B'} \vec{f} \rho dV = \\ &= \int_{B'} \vec{g}' dm = m \vec{g}' \end{aligned}$$

$$\boxed{\begin{aligned} \vec{F}_g &= m \vec{g} \\ \vec{f}' &= \vec{g}' \end{aligned}}$$

$$\vec{N}'(P_0') = \int_{B'} (P' - P_0') \times \vec{f} dm = \int_{B'} (P - P_0') \times \vec{g} dm = \left(\int_{B'} (P - P_0') dm \right) \times \vec{g} =$$

masa red

$$= (P_*' - P_0') \times \vec{g}' m = (P_*' - P_0') \times \vec{F}'$$

Sila teže (homogeno telo = neodvisna od položaja) moramo nadomestiti s točkovno silo, ki ima prijemališče v masnemredišču. rezultanto

$$m \vec{a}_x' = m \vec{g}' + \vec{S}' + \vec{F}'$$

$$\boxed{\begin{aligned} \vec{F}' &= F \vec{e}_1 \\ \vec{S}' &= S \vec{e}_2 \end{aligned}}$$

$$\vec{e}_1 \quad m \ddot{x} = mg \sin \alpha - F$$

$$\vec{e}_2 \quad m \ddot{y} = -mg \cos \alpha + S \quad (\text{simetričnost?})$$

SIST. ENACB je diagonalna, $J_{13}, J_{23}, \dots$ so 0!

$$J_{33} \ddot{\varphi} = N_3(P_*)$$

$$\vec{N}'(P_*) = \underbrace{0}_{\vec{0}} \times m \vec{g} + \underbrace{(-R \vec{e}_2 \times \vec{S}')}_{\vec{0}} + \underbrace{(-R \vec{e}_2 \times \vec{F}')}_{R F \vec{e}_3}$$

$$\begin{aligned} &\parallel R F \vec{e}_2 \times \vec{e}_1 \\ &\parallel R F \vec{e}_3 \end{aligned}$$

dobimo

$$\Rightarrow \ddot{\varphi} = N_3(P_*) = -RF$$

+ točke 2

3 dimenzionalne enačbe, 2 5
mehanikami.kinematični pogoji: $y=R$ 4. enačba• kotajenje: telesna točka dotikališča
ima hitrost podlage = 0

$$\dot{P}' = \dot{P}_0' + \vec{\omega}' \times (P' - P_0') = \vec{0}$$

 P' dotikališči, $P_0' = P_*$

$$P' - P_0' = -R\vec{e}_2$$

$$\vec{0} = \dot{x}\vec{e}_1 + \dot{\varphi}\vec{e}_3 \times (-R\vec{e}_2) = \dot{x}\vec{e}_1 + R\dot{\varphi}\vec{e}_1$$

$$\boxed{\dot{x} + R\dot{\varphi} = 0} \quad 5. \text{ enačba}$$

$$\text{Rešimo sistem: } \left(\frac{1}{2}mR^2\right)\left(-\frac{\ddot{x}}{R}\right) = -RF \Rightarrow F = \frac{1}{2}m\ddot{x}$$

$$\Rightarrow m\ddot{x} = mg \sin \alpha - \frac{1}{2}m\ddot{x} \Rightarrow \frac{3}{2}\ddot{x} = g \sin \alpha$$

$$x = \left(\frac{1}{3}g \sin \alpha\right)t^2 + c_1 t + c_2$$

$$\underline{y = R}$$

$$\text{vayj opustitno: } c_1 = c_2 = 0$$

$$\Rightarrow \underline{x = \frac{1}{3}g \sin \alpha t^2}$$

$$S = mg \cos \alpha$$

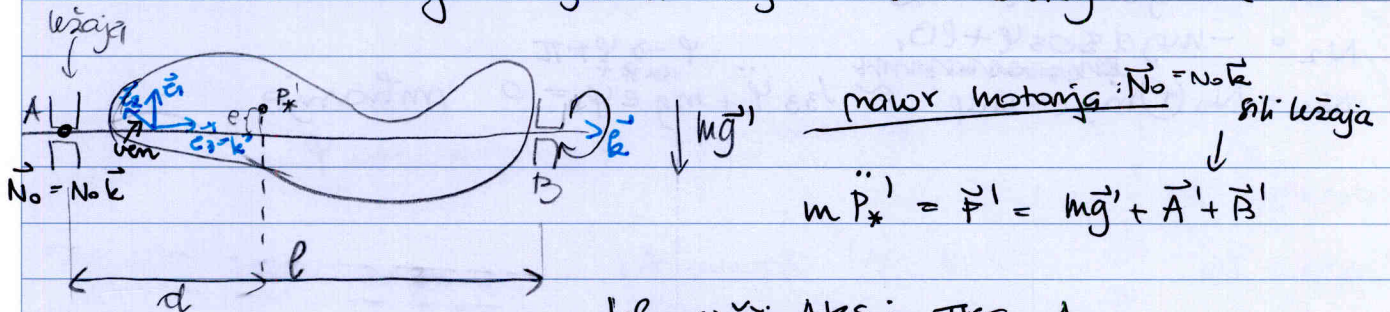
$$\underline{F = \frac{1}{3}mg \sin \alpha}$$

koef. trenja

Podlaga je sposobna reakcije F , če je $F \leq kS$

$$\frac{1}{3}mg \sin \alpha \leq k \cdot mg \cos \alpha$$

$$\underline{\tan \alpha \leq 3k}$$

PRIMER 2 vrtenje ekscentričnega telesa okrog stalne osi

pravor motorja: $\vec{N}_0 = N_0 \vec{k}$ sili ležaja

$$m \ddot{P}_* = \vec{F}' = m\vec{g}' + \vec{A}' + \vec{B}'$$

Izhodišči AKS in TKS v A.

Enačba gibanja masnega središča za opazovalca tks ~~zapisano~~
zapisano v AKS:

0 izh. uaki

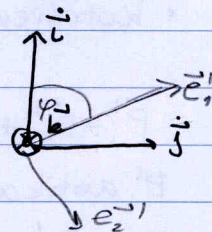
$$\vec{0} = m\vec{g}' + \vec{A}' + \vec{B}' + (-m\vec{a}_0' - m\vec{\omega}' \vec{r}' - m\vec{\omega}' \times (\vec{\omega}' \times \vec{r}')) - \underbrace{2m\vec{\omega}' \times \vec{v}_m'}_{\substack{\text{"} \\ \text{Larmire}}}$$

$$\bullet \vec{\omega}' \times \vec{r}' = \dot{\varphi} \vec{k} \times (d\vec{e}_1' + \vec{e}_1' \cdot \vec{e}) = \dot{\varphi} e \vec{e}_3' + \vec{e}_1' \times \vec{e} = \dot{\varphi} e \vec{e}_2'$$

$$\bullet \vec{\omega}' \times (\vec{\omega}' \times \vec{r}') \stackrel{\substack{\text{"} \\ \vec{e}_3'}}{=} \dot{\varphi}^2 \vec{e}_3' \times (\dot{\varphi} e \vec{e}_2') = -e \dot{\varphi}^2 \vec{e}_1'$$

$$\bullet \vec{\omega}' \times \vec{r}' = \ddot{\varphi} e \vec{e}_2'$$

$$\bullet \vec{g}' = -g \vec{e}_1'$$



$$\vec{r}' = \cos\varphi \vec{e}_1' - \sin\varphi \vec{e}_2'$$

$$\Rightarrow \vec{g}' = -g \cos\varphi \vec{e}_1' + g \sin\varphi \vec{e}_2'$$

$$\vec{e}_1': 0 = -mg \cos\varphi + A_1 + B_1 + m e \dot{\varphi}^2 \quad (1)$$

$$\vec{e}_2': 0 = +mg \sin\varphi + A_2 + B_2 - m e \ddot{\varphi} \quad (1)$$

$$\vec{e}_3': 0 = A_3 + B_3$$

(3) $\rightarrow$ sistem jednač za gibanje mase na pred. sč.

venio:

$$N_1 = J_{13} \cdot \ddot{\varphi} - \dot{\varphi}^2 J_{23}$$

$$N_2 = \ddot{\varphi} J_{23} + \dot{\varphi}^2 J_{13}$$

$$N_3 = J_{33} \ddot{\varphi}$$

$$\vec{N}' = (\vec{P}_x' - A) \times m\vec{g}' + (A - A) \times \vec{A}' + (B - A) \times \vec{B}' + \vec{N}_0'$$

$$\bullet (\vec{P}_x' - A) \times m\vec{g}' = mg(d\vec{e}_3' + e\vec{e}_1') \times (-\cos\varphi \vec{e}_1' + \sin\varphi \vec{e}_2') =$$

$$= mg(-d \cos\varphi \vec{e}_2' - d \sin\varphi \vec{e}_1' + e \sin\varphi \vec{e}_3')$$

$$\bullet (B - A) \times \vec{B}' = \vec{e}_3' \times (B_1 \vec{e}_1' + B_2 \vec{e}_2' + B_3 \vec{e}_3') =$$

$$= \ell B_1 \vec{e}_2' - \ell B_2 \vec{e}_1'$$

$$\Rightarrow \vec{N} = (-mg d \sin\varphi - \ell B_2) \vec{e}_1' + (-mg \cos\varphi d + \ell B_1) \vec{e}_2' + (mg e \sin\varphi + N_0) \vec{e}_3'$$

$$\vec{N}' = Q \vec{N} \quad \vec{e}_i' = Q \vec{e}_i$$

$$\Rightarrow N_1 = -mg d \sin\varphi - \ell B_2$$

$$N_2 = -mg d \cos\varphi + \ell B_1$$

$$N_3 = N_0 + mg e \sin\varphi \approx J_{33} \ddot{\varphi} + mg e \varphi = 0 \quad \text{mhanje}$$

prigledimo

$\varphi \rightarrow \varphi + \pi$
dobio ven φ

dobio B_1

dobio B_2

$\Rightarrow$ iz (1) dobio A_1

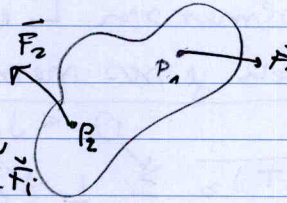
iz (2) dobio A_2

PRIMER 3 Prosta vrtavka = npr. Zemlja brez sile pona in premetov
 $\vec{F}' = \vec{0}$ in $\vec{N}' = \vec{0}$ *def. proste vrtavke*

$\mathcal{F} = \{ (P_1, \vec{F}_1), \dots, (P_n, \vec{F}_n) \}$ sistem sil

$\vec{N}(\vec{P}_0) = \sum_{i=1}^n (\vec{P}_i - \vec{P}_0) \times \vec{F}_i$

$\vec{N}(\vec{P}_0) \stackrel{!}{=} \sum_{i=1}^n \underbrace{(\vec{P}_i - \vec{P}_0)}_{(\vec{P}_i - \vec{P}_0) + (\vec{P}_0 - \vec{P}_0)} \times \vec{F}_i = \vec{N}(\vec{P}_0) + (\vec{P}_0 - \vec{P}_0) \times \sum_{i=1}^n \vec{F}_i$



$\Rightarrow$ trditev Če je rezultanta sil enaka nič, je mason tega sistema sil neodvisna od mase.

$$m \ddot{\vec{P}}_0 = \vec{0}$$

* $0 = J_1 \omega_1 - \omega_2 \omega_3 (J_2 - J_3)$

** $0 = J_2 \omega_2 - \omega_3 \omega_1 (J_3 - J_1)$

*** $0 = J_3 \omega_3 - \omega_1 \omega_2 (J_1 - J_2)$

} enačim. enačbe

Def: $\omega_1 = \omega_2 = \omega_3 = 0 \Rightarrow$ enakomerno vrtenje

možne rešitve: $\omega_1 = \omega_2 = 0$

ali $\omega_1 = \omega_3 = 0$

ali $\omega_2 = \omega_3 = 0$

$\Rightarrow$ vsaj eno osi gibanja osti

samo

$\Rightarrow$ trditev Prosta vrtavka se lahko enakomerno vrtilno gibanje okoli ene od treh osi vrtenja.

1 kin. energ.

$$\frac{dT}{dt} = 0, \quad T \text{ konst.}$$

$$\vec{N}' = 0 \Rightarrow \vec{L}' = \text{konstantna}, \quad \text{velikost } \sqrt{\vec{L}' \cdot \vec{L}'} = \text{konst.}$$

$$\vec{L}' = \Omega \vec{L} = \Omega (J \vec{\omega})$$

$$|\vec{L}'| = \text{konst} \Rightarrow |\vec{L}| = \text{konst}$$

2 konstante gibanja: $\bullet \vec{L}$

$$\vec{L} = (J_1 \omega_1) \vec{e}_1 + (J_2 \omega_2) \vec{e}_2 + (J_3 \omega_3) \vec{e}_3$$

$$|\vec{L}|^2 = \underbrace{(J_1 \omega_1)^2}_{L_1^2} + \underbrace{(J_2 \omega_2)^2}_{L_2^2} + \underbrace{(J_3 \omega_3)^2}_{L_3^2}$$

Energija T

$$T = \frac{1}{2} \vec{\omega} \cdot J \vec{\omega}$$

$$= \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} J_2 \omega_2^2 + \frac{1}{2} J_3 \omega_3^2$$

$$T = \frac{L_1^2}{2J_1} + \frac{L_2^2}{2J_2} + \frac{L_3^2}{2J_3} \quad / : T$$

$$L_1^2 + L_2^2 + L_3^2 = L^2$$

$$\frac{L_1^2}{2J_1} + \frac{L_2^2}{2J_2} + \frac{L_3^2}{2J_3} = 1$$

sfera
Bimetor
elipsoid

Komponente vrtnih količin ležijo na preseku sfere s polmerom L in elipsoidom s polosmi $\sqrt{2TJ_1}, \sqrt{2TJ_2}, \sqrt{2TJ_3}$.

Preseka je nesimetričen:

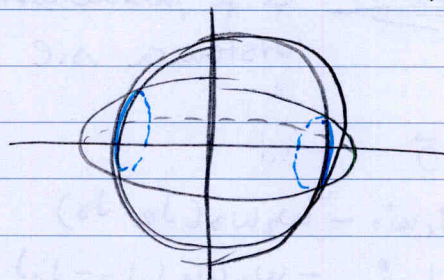
$$0 < J_1 \leq J_2 \leq J_3 \quad \text{kruga zunaj}$$

$$\sqrt{2TJ_3} \geq L \geq \sqrt{2TJ_1}$$

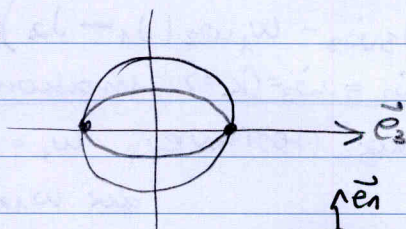
$$1 = \frac{L_1^2}{J_1} + \frac{L_2^2}{J_2} + \frac{L_3^2}{J_3} \leq \frac{1}{2TJ_1} (L_1^2 + L_2^2 + L_3^2) = \frac{L^2}{2TJ_1}$$

→ enako za to!

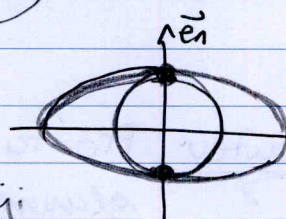
$J_1 < J_2 < J_3$
kruga znotraj



Enakomerna rotacija okrog $\vec{e}_3$:
je stabilna.



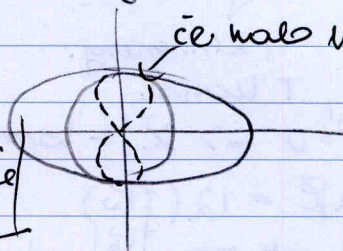
Enakomerna rot. okrog $\vec{e}_1$:
je stabilna (u nasredino
majhno motnjo)



Enakom. rot. okrog $\vec{e}_2$: preide v 2 krivulji
je nestabilna

če halo povečamo

kerne: sn, cn, dn = Jacobijev
eliptične funkcije



Prosta simetrična vrtenja

simetrična $\Rightarrow J_1 = J_2$

$$\leadsto *** J_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{konst}$$

$$J_1 \dot{\omega}_1 = \omega_2 \omega_3 (J_1 - J_3) \Rightarrow \dot{\omega}_1 = \frac{\omega_2 (J_1 - J_3)}{J_1} \omega_2$$

$$\dot{\omega}_1 = -\Omega \omega_2$$

$$\dot{\omega}_2 = -\Omega \omega_1$$

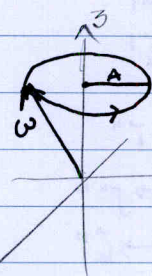
$$\ddot{\omega}_1 = -\Omega \dot{\omega}_2 = -\Omega (-\Omega \omega_1) \Rightarrow \ddot{\omega}_1 + \Omega^2 \omega_1 = 0 \leadsto$$

$$\left(\begin{aligned} \dot{\underline{\omega}} &= \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \underline{\omega} & \omega = e^{[T]} \end{aligned} \right) ?$$

Enačba krovnega vrtilanja:

$$\Rightarrow \begin{cases} \omega_1 = A \cos(\Omega t - \delta) \\ \omega_2 = A \sin(\Omega t - \delta) \\ \omega_3 = \text{konst.} \end{cases} \quad \text{rešitev}$$

TKS

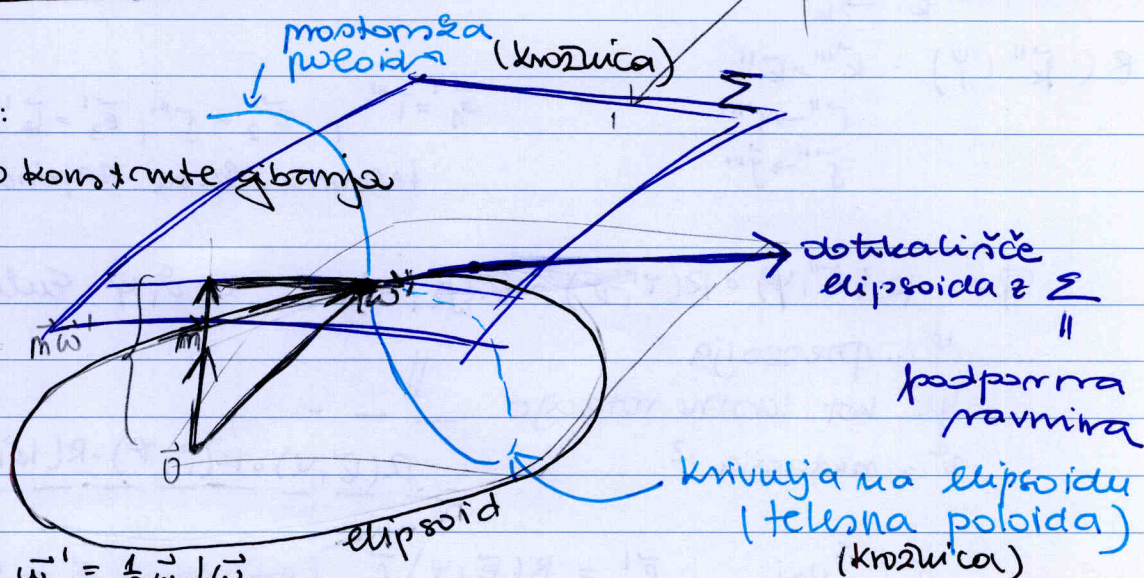


$\vec{\omega}$ meršina
(kroži) okrog
osi 3
frekvenco Ω .

Slika v AKS:

$\vec{x}', T$ so konstante gibanja
 $\vec{m} = \frac{\vec{x}'}{|\vec{x}'|}$

konst = $m\vec{\omega}$



$$T = \frac{1}{2} \vec{\omega}' \cdot \vec{\omega}' = \frac{1}{2} \vec{\omega} \cdot \vec{\omega}$$

$$T = \frac{1}{2} \vec{x}' \cdot \vec{\omega}' = \frac{1}{2} \vec{m} \cdot |\vec{x}'| \cdot \vec{\omega}' \Rightarrow m \vec{\omega}' = \frac{2T}{|\vec{x}'|} = \text{konst.}$$

$\vec{\omega}'$ leži na Binetovem elipsoidu $T = \frac{1}{2} \vec{\omega}' \cdot \vec{\omega}'$.

$$F = T - \frac{1}{2} \vec{\omega}' \cdot \vec{\omega}'$$

$$\text{grad } F = \left(\frac{\partial F}{\partial \vec{\omega}'} \right)^T = -\vec{\omega}' = -\vec{x}', \quad \vec{m} \text{ je normala na elipsoid}$$

Elipsoid se kotiti po podporni ravnini in opiše neklo krivuljo

primer: žaucja

$$\frac{J_1 - J_3}{J_1} \approx \frac{1}{306}$$

$$\omega_3 = \frac{2\pi}{24} \text{ kr (dnevna rot.)}$$

amplituda nihanja $\approx 10 \text{ m}$

izmenjevo $\updownarrow = 420 \text{ dni}$

$$\Rightarrow \omega \Omega = \frac{J_1 - J_3}{J_1} \omega_3$$

$T \approx 306 \text{ dni} \approx 10 \text{ mes.}$

Akta

$$T \approx 306 \text{ dni}$$

$$\frac{1.2\pi}{(306 \cdot 24 \text{ ur})} = \frac{2\pi}{T} \approx \Omega$$

Eulerjevi koti

rotacija
 $R(\vec{k}, \varphi) : \vec{i} \rightarrow \vec{i}'$

$$\vec{j} \rightarrow \vec{j}'$$

$$\vec{k} = \vec{k}'$$

$R(\vec{i}', \vartheta) : \vec{i}' \rightarrow \vec{i}''$

$$\vec{j}' \rightarrow \vec{j}''$$

$$\vec{k}' \rightarrow \vec{k}''$$

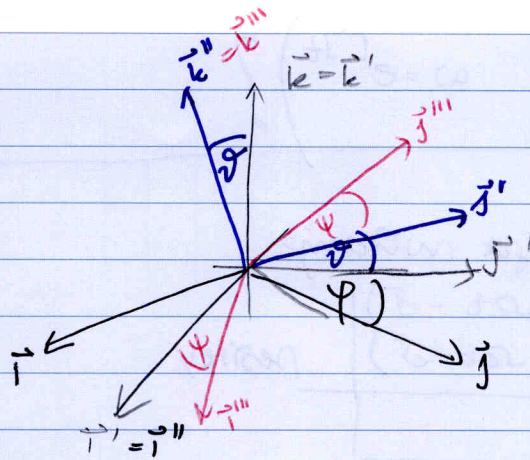
$R(\vec{k}'', \psi) : \vec{k}'' \rightarrow \vec{k}'''$

$$\vec{i}'' \rightarrow \vec{i}'''$$

$$\vec{j}'' \rightarrow \vec{j}'''$$

$$\vec{e}_1' = \vec{i}''', \vec{e}_2' = \vec{j}''', \vec{e}_3' = \vec{k}'''$$

telesna baza zapirana u AKS



$$\mathcal{R} = R(\vec{k}'', \psi) \circ R(\vec{i}', \vartheta) \circ R(\vec{k}, \varphi) \quad \varphi, \vartheta, \psi \text{ Eulerjevi koti}$$

φ ... precesija

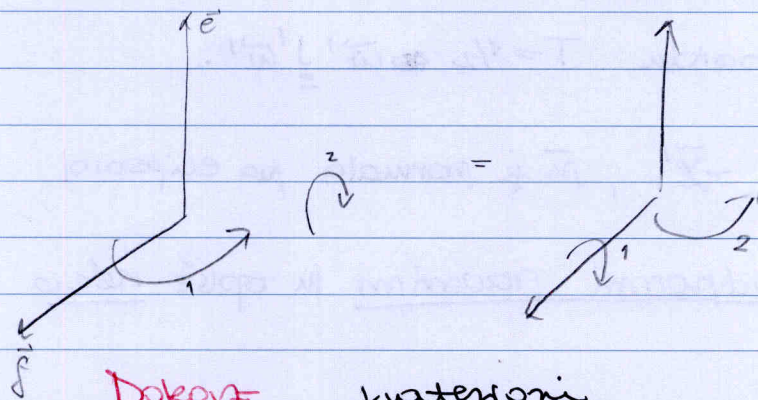
ψ ... kot lastne rotacije

ϑ ... nutacija?

$$\parallel \quad \underline{\underline{R(\vec{e}, \varphi) \circ R(\vec{i}', \vartheta) \circ R(\vec{k}'', \psi)}}$$

Lemma

$\vec{e}, \vec{f}$ osi, $\vec{f}' = R(\vec{e}, \varphi) \vec{f}$ (rot. okrog $\vec{e}$ za φ vektorja $\vec{f}$)
 $R(\vec{f}', \vartheta) \circ R(\vec{e}, \varphi) = R(\vec{e}, \varphi) \circ R(\vec{f}, \vartheta)$.



= lahko obrnemo
 vrstni red
 rotacij

Dokaz kvaternioni

$$q_1 = R(\vec{e}, \varphi), q_2 = R(\vec{f}, \vartheta), q_2' = R(\vec{f}', \vartheta)$$

$$\underline{\underline{q_2' q_1 = q_1 q_2}}$$

$$\vec{f}' = R(\vec{e}, \varphi) \vec{f} = \cos \varphi \vec{f} + (\vec{f} \cdot \vec{e})(1 - \cos \varphi) \vec{e} + \sin \varphi (\vec{e} \times \vec{f})$$

$$q_2' q_1 = \left(\cos \frac{\vartheta}{2} + \sin \frac{\vartheta}{2} \vec{f}' \right) \left(\cos \frac{\varphi}{2} + \sin \frac{\varphi}{2} \vec{e} \right) =$$

$$\begin{aligned} & \left(\cos \frac{\varphi}{2} \cos \frac{\varphi}{2} - (\vec{e} \cdot \vec{f}) \sin \frac{\varphi}{2} \sin \frac{\varphi}{2} \right) + \left(\cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \vec{e} + \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \vec{f} + \sin \frac{\varphi}{2} \sin \frac{\varphi}{2} (\vec{f} \times \vec{e}) \right) \\ \Rightarrow \vec{e} \cdot \vec{f}' &= (\vec{e} \cdot \vec{f}) (\cos \varphi + (1 - \cos \varphi)) = \vec{e} \cdot \vec{f} \quad \left((\vec{e} \times \vec{f}) \times \vec{e} = \vec{f} - (\vec{e} \cdot \vec{f}) \vec{e} \right) \\ &= \left(\vec{e} \cdot \vec{f} \right) \cos \varphi + \vec{e} \left(\cos \frac{\varphi}{2} \sin \frac{\varphi}{2} + (\vec{f} \cdot \vec{e}) (1 - \cos \varphi) \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} - \right. \\ & \quad \left. - (\vec{e} \cdot \vec{f}) \sin \frac{\varphi}{2} \sin \frac{\varphi}{2} \sin \varphi \right) = \dots \end{aligned}$$

okrog znat. osi

$$R(\vec{e}_n, \dots, \vec{e}_1; \varphi_n, \dots, \varphi_1) = R(\vec{e}_n^{(n)}, \varphi_n) \circ \dots \circ R(\vec{e}_2^{(2)}, \varphi_2) \circ R(\vec{e}_1^{(1)}, \varphi_1)$$

$$\vec{e}_i^{(1)} = R(\vec{e}_1, \varphi_1) \vec{e}_i \quad , \quad i = 2, \dots, n$$

$$\vec{e}_i^{(2)} = R(\vec{e}_2, \varphi_2) \vec{e}_i^{(1)} \quad , \quad i = 3, \dots, n$$

⋮

$$\vec{e}_i^{(n)} = R(\vec{e}_n, \varphi_n) \vec{e}_i^{(n-1)} \quad , \quad i = 1, \dots, n$$

induktivno

Eulerjeva notacija

IZREK $R(\vec{e}_n, \dots, \vec{e}_1; \varphi_n, \dots, \varphi_1) = R(\vec{e}_1, \varphi_1) \circ R(\vec{e}_2, \varphi_2) \circ \dots \circ R(\vec{e}_n, \varphi_n)$

DOKAZ

$n=1$ očitno velja

($n=2$ velja po lemi)

$$n \rightarrow n+1: R(\vec{e}_n, \dots, \vec{e}_1; \varphi_n, \dots, \varphi_1) =$$

$$= R(\vec{e}_n', \dots, \vec{e}_2'; \varphi_n, \dots, \varphi_2) \circ R(\vec{e}_1, \varphi_1) =$$

$$= R(\vec{e}_2', \varphi_2) \circ \dots \circ R(\vec{e}_n', \varphi_n) \circ R(\vec{e}_1, \varphi_1) =$$

po ind. predp.

$$= R(\vec{e}_1, \varphi_1) \circ R(\vec{e}_2', \varphi_2) \circ \dots \circ R(\vec{e}_n', \varphi_n)$$

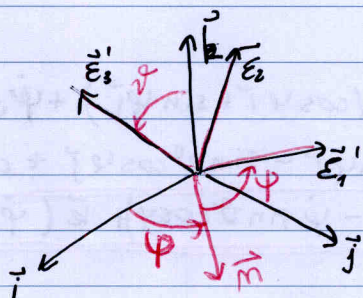
po lemi

IZREK

Vsako rotacijo, ki ni rotacija okrog $\vec{k}$, lahko enolično zapišemo z Eulerjevimi koti

$$\varphi \in [0, 2\pi), \psi \in [0, 2\pi), \vartheta \in [0, \pi).$$

DOKAZ



$$R \text{ ni rotacija okrog } \vec{k} \Rightarrow \vec{e}_3' \nparallel \vec{k}$$

$$\vec{m} \perp \vec{e}_3', \vec{k}$$

inolicnost

Recimo, da $\vec{r}$ završava:

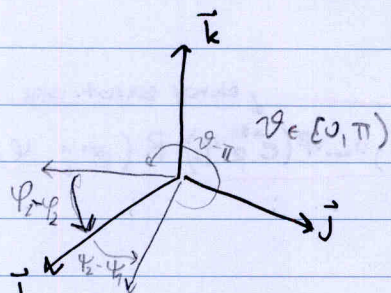
$$R = R(\psi_1, \vec{k}) \circ R(\vartheta_1, \vec{i}) \circ R(\psi_1, \vec{k}) = R(\psi_2, \vec{k}) \circ R(\vartheta_2, \vec{i}) \circ R(\psi_2, \vec{k})$$

$$R(\vec{i}, \vartheta_1) = R(\vec{k}, \psi_2 - \psi_1) \circ R(\vartheta_2, \vec{i}) \circ R(\vec{k}, \psi_2 - \psi_1)$$

$$R(\vec{i}, \vartheta_1) \vec{i} = \vec{i} = R(\vec{k}, \psi_2 - \psi_1) \circ R(\vartheta_2, \vec{i}) \circ R(\vec{k}, \psi_2 - \psi_1) \vec{i}$$

$$\vec{i} = \hat{Q} \vec{i}$$

rotacija okrog $\vec{i}$



$$\Rightarrow \psi_2 = \psi_1$$

$$\psi_2 = \psi_1$$

menjavo izimativ i, razen če

$$R = R_3 R_2 R_1$$

$$Q^T Q \vec{a} = \vec{\omega} \times \vec{a}$$

$$Q \vec{a} = Q(\vec{\omega} \times \vec{a}) = (Q\vec{\omega}) \times (Q\vec{a}) = \vec{\omega}' \times Q\vec{a}$$

velja: $Q' = \frac{d}{dt}(Q_3 Q_2 Q_1) = Q_3' Q_2 Q_1 + Q_3 Q_2' Q_1 + Q_3 Q_2 Q_1'$

$$Q' \vec{a} = Q_3' (Q_2 Q_1 \vec{a}) + Q_3 (Q_2' (Q_1 \vec{a})) + Q_3 (Q_2) (Q_1' \vec{a}) =$$

$$= \vec{\omega}_3' \times Q_3 (Q_2 Q_1 \vec{a}) + Q_3 (\vec{\omega}_2' \times Q_2 Q_1 \vec{a}) + Q_3 Q_2 (\vec{\omega}_1' \times Q_1 \vec{a}) =$$

$$\Rightarrow Q' \vec{a} = (\vec{\omega}_3' + Q_3 \vec{\omega}_2' + Q_3 Q_2 \vec{\omega}_1') \times Q \vec{a}$$

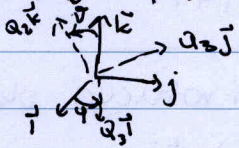
$$\vec{\omega}' = \vec{\omega}_3' + Q_3 \vec{\omega}_2' + Q_3 Q_2 \vec{\omega}_1'$$

$$Q_1 = R(\vec{k}, \psi) \quad Q_2 = R(\vec{i}, \vartheta), \quad Q_3 = R(\vec{k}, \psi)$$

$$\vec{\omega}_1' = \dot{\psi} \vec{k}, \quad \vec{\omega}_2' = \dot{\vartheta} \vec{i}, \quad \vec{\omega}_3' = \dot{\psi} \vec{k}$$

rotacije:

$$Q_3 \vec{i} = \cos \psi \vec{i} + \sin \psi \vec{j}$$



$$Q_2 \vec{k} = -\sin \vartheta \vec{j} + \cos \vartheta \vec{k}$$

$$Q_3 \vec{j} = -\sin \psi \vec{i} + \cos \psi \vec{j}$$

↓

$$\vec{\omega}' = \dot{\psi} \vec{k} + \dot{\vartheta} Q \vec{i} + \dot{\psi} Q_3 Q_2 \vec{k} = \dot{\psi} \vec{k} + \dot{\vartheta} (\cos \psi \vec{i} + \sin \psi \vec{j}) + \dot{\psi} Q_3 (-\sin \vartheta \vec{j} + \cos \vartheta \vec{k}) =$$

$$= \dot{\psi} \vec{k} + \dot{\vartheta} (\cos \psi \vec{i} + \sin \psi \vec{j}) + \dot{\psi} (\sin \vartheta \sin \psi \vec{i} - \sin \vartheta \cos \psi \vec{j} + \cos \vartheta \vec{k}) =$$

$$= \vec{i} (\dot{\vartheta} \cos \psi + \dot{\psi} \sin \vartheta \sin \psi) + \vec{j} (\dot{\vartheta} \sin \psi - \dot{\psi} \sin \vartheta \cos \psi) + \vec{k} (\dot{\psi} + \dot{\vartheta} \cos \vartheta)$$

vektor rotacijske hitrosti

v prostorski bazi

Iščemo $\vec{\omega}'$ v telesni bazi:

$$\vec{\omega}' = Q \vec{\omega}$$

1. Poiščemo $\vec{\omega}$ v montonzi bazi $\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}$

2. $\vec{\omega}' = Q \vec{\omega} = \omega_1 \vec{i}' + \omega_2 \vec{j}' + \omega_3 \vec{k}'$ To je iskani zapis.

$$\begin{matrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{matrix}$$

Velja: $\vec{\omega}$ je osni vektor rotacije $Q \Leftrightarrow -\vec{\omega}$ je vektor kotne hitrosti rotacije Q^T .

$$\vec{\omega} = -\vec{\omega}' (R(\vec{k}, \vartheta, \psi; \psi, \vartheta, \psi)^T).$$

Trditve:

$$\begin{aligned} R(\vec{k}, \vec{i}, \vec{k}; \psi, \vartheta, \psi)^T &= (R(\vec{k}, \psi) \circ R(\vec{i}, \vartheta) \circ R(\vec{k}, \psi))^T = \\ &= R(\vec{k}, \psi)^T \circ R(\vec{i}, \vartheta)^T \circ R(\vec{k}, \psi)^T = \\ &= R(\vec{k}, -\psi) \circ R(\vec{i}, -\vartheta) \circ R(\vec{k}, -\psi) = \underline{R(\vec{k}, \vec{i}, \vec{k}; -\psi, -\vartheta, -\psi)}. \end{aligned}$$

$$\begin{aligned} \vec{\omega} &= - \left[\vec{i} (-\dot{\vartheta} \cos \psi - \dot{\psi} (-\sin \vartheta) (-\sin \psi)) + \vec{j} (\dot{\vartheta} \sin \psi + \dot{\psi} \sin(-\vartheta) \cos \psi) + \right. \\ &\quad \left. + \vec{k} (-\dot{\psi} - \dot{\vartheta} \cos \vartheta) \right] = \\ &= \vec{i} (\dot{\vartheta} \cos \psi + \dot{\psi} \sin \vartheta) + \vec{j} (-\dot{\vartheta} \sin \psi + \dot{\psi} \sin \vartheta \cos \psi) + \vec{k} (\dot{\psi} + \dot{\vartheta} \cos \vartheta). \end{aligned}$$

$$\vec{\omega}' = \vec{e}_1' (\dot{\vartheta} \cos \psi + \dot{\psi} \sin \vartheta) + \vec{e}_2' (-\dot{\vartheta} \sin \psi + \dot{\psi} \sin \vartheta \cos \psi) + \vec{e}_3' (\dot{\psi} + \dot{\vartheta} \cos \vartheta).$$

vektor kotne hitrosti v telesni bazi

$$N_1 = J_1 \dot{\omega}_1 - \omega_2 \omega_3 (J_2 - J_3).$$

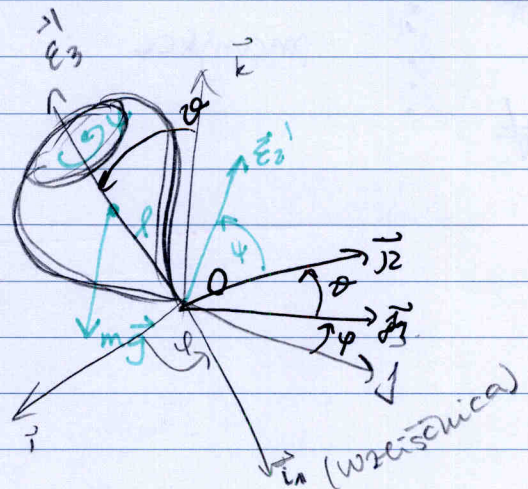
Simetrična vrtavka

$$J_1 = J_2$$

$$\begin{aligned} \vec{N}' &= \ell \vec{e}_3' \times m \vec{g} = -mg\ell \vec{e}_3' \times \vec{k} = \\ &= +mg\ell \sin \vartheta \vec{i}_1 \end{aligned}$$

$$\vec{N}' \cdot \vec{k} = 0$$

$$\vec{N}' = \frac{d\vec{L}'}{dt}$$



$$\vec{N}' \cdot \vec{k} = \frac{d}{dt}(\vec{L}' \cdot \vec{k}) = 0 \Rightarrow \vec{L}' \cdot \vec{k} = \text{konstanta}$$

komponenta vrt. količine gibanja (k) je konstanta.

$$\vec{N} \cdot \vec{E}_3' = 0; \vec{E}_3' \text{ odvisen od } \vartheta \Rightarrow \vec{N}_3 = 0 \xrightarrow[\text{simetr.}]{J_1=J_2} J_3 \dot{\varphi} = 0 \Rightarrow \omega_3 = \text{konst}$$

Simetrična vrtavka ima 3 domst. gibanja: ω_3 , $\vec{L}' \cdot \vec{k}$, $T + mgl \cos \vartheta$.
energija

$$\begin{aligned} \vec{L}' \cdot \vec{k} &= (\underline{J} \underline{\omega}') \cdot \vec{k} = (\underline{Q} \underline{J} \underline{\omega}) \cdot \vec{k} = (J_1 \omega_1 \vec{e}_1' + J_2 \omega_2 \dots) \cdot \vec{k} \\ \underline{J}' &= \underline{Q} \underline{J} \underline{Q}^T & \underline{J} \underline{\omega} &= J_1 \omega_1 \vec{e}_1 + J_2 \omega_2 \vec{e}_2 + J_3 \omega_3 \vec{e}_3 \\ \underline{\omega}' &= \underline{Q} \underline{\omega} \end{aligned}$$

$$\begin{aligned} \vec{k} &= \sin \vartheta \vec{e}_2 + \cos \vartheta \vec{e}_3' = \sin \vartheta (\sin \psi \vec{e}_1' + \cos \psi \vec{e}_2') + \cos \vartheta \vec{e}_3' \\ \Rightarrow \vec{L}' \cdot \vec{k} &= J_1 \omega_1 \sin \vartheta \sin \psi + J_2 \omega_2 \sin \vartheta \cos \psi + J_3 \omega_3 \cos \vartheta \end{aligned}$$

... vstavimo $\omega_1, \omega_2, \omega_3$ iz vektorja kotne hitrosti v telesni bazi...
in upoštevamo $J_1 = J_2$

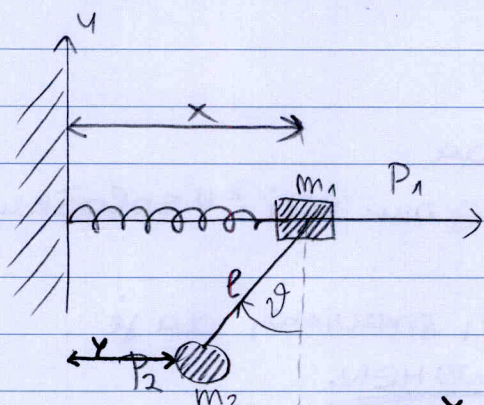
$$\begin{aligned} \vec{L}' \cdot \vec{k} &= J_1 \sin^2 \vartheta \dot{\varphi} + J_3 (\dot{\psi} + \dot{\varphi} \cos \vartheta) \cos \vartheta = \dot{\varphi} (J_1 \sin^2 \vartheta + J_3 \cos^2 \vartheta) + \dot{\psi} J_3 \cos \vartheta \\ &= J_1 \dot{\varphi} + \text{nekaj konstanta} \\ \therefore \text{ne } \omega_3, E \dots \text{manjka} \end{aligned}$$

Hitra vrtavka $\Leftrightarrow \frac{1}{2} J_3 \omega_3^2 \gg mgl$

~~manjka~~ $\vdots$
 $\neq$ $\vdots$
manjka

2. LAGRANGE-EVA MEHANIKA

GENERALIZIRANE KOORDINATE



$$\underline{q} \in \mathbb{R}^n$$

$$P_1 = (x_1, y_1)$$

$$P_2 = (x_2, y_2)$$

$$P_1 = (x_1, 0)$$

+ vez

$$l = \sqrt{(x_1 - x_2)^2 + y_2^2}$$

x, y - sistem ima dve prostostni stopnji
generalizirani koordinati

Če mi vezi, pravimo, da so koordinate neodvisne.

VEZI

$f(\underline{q}, t) = 0$ - geometrijska vez

$$f(\underline{q}) = 0 \quad + \quad \underline{q} \in \mathbb{R}^n$$

Prostoru dopustnih koordinat pravimo konfiguracijski prostor
 $\underline{q} \in \mathbb{R}^n + \text{vezi}$

$$f_i(\underline{q}) = 0 \quad \text{za} \quad i = 1, \dots, m, \quad m < n$$

$$\underline{f} = (f_1, \dots, f_m)$$

$$\text{rang} \left(\frac{\partial \underline{f}}{\partial \underline{q}} \right) = m \Rightarrow \text{konfiguracijski prostor je } m\text{-m razsežna plošča (mnogokotnik)}$$

Vezi, ki je neodvisna od časa pravimo SKLERONOMNA vez.

Vezi, ki je odvisna od časa pravimo REONOMNA vez.

$$f(\underline{q}, t) = 0 \quad (\text{npr. gibanje po elipsoidu, ki se deformira v odvisnosti od časa})$$

vezi mvega reda: $f(\underline{q}, \dot{\underline{q}}, t) = 0$ (vez je funkcijska zveza med splošnimi hitrostmi)

vezi višjega reda: $f(\underline{q}, \dot{\underline{q}}, \dots, \frac{d^k \underline{q}}{dt^k}, t) = 0$

Odvod geometrijske vezi trajektorije $\underline{q} = \underline{q}(t)$ nam da vez mvega reda.

$$0 = \frac{d}{dt} f(q, t) = \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial q} \right)^T \dot{q} + \frac{\partial f}{\partial t}$$

Kvazilinearna vez mvega reda je oblike:

$$a(q, t) \cdot \dot{q} + a_0(q, t) = 0$$

$$a_i(q, t) \dot{q}_i + a_0(q, t) = 0$$

Omejili se bomo ^{na} ~~na~~ kvazilin. veri mvega reda

Ali je vsaka kvazilin. vez 1. reda odvod geom. veri? V splošnem **NE!**

Če je sistem veri mvega reda integrabilen, pravimo, da je sistem HOLONOMEN, drugače je NEHOLONOMEN.

$$\left. \begin{aligned} a_i(q, t) \dot{q}_i + a_0(q, t) &= 0 \\ a_i(q, t) &= \frac{\partial f}{\partial q_i} \quad ; \quad q_0 = t \end{aligned} \right\} \sum_{i=0}^n a_i \dot{q}_i = 0$$

$$\text{tako je } \lambda \cdot a_i = \frac{\partial f}{\partial q_i} \quad ; \quad \lambda \neq 0$$

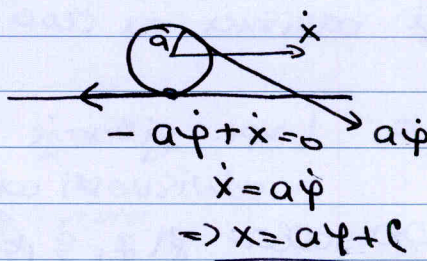
$$\Rightarrow \frac{\partial(\lambda a_i)}{\partial q_j} = \frac{\partial(\lambda a_j)}{\partial q_i}$$

IZREK (Frobenius)

Vez $a_i(q, t) \cdot \dot{q}_i + a_0(q, t) = 0$ je integrabilna $\Leftrightarrow$ ko $\exists$ neničelni množitelj $\lambda = \lambda(q, t)$ tako, da je

$$\frac{\partial(\lambda a_i)}{\partial q_j} = \frac{\partial(\lambda a_j)}{\partial q_i}.$$

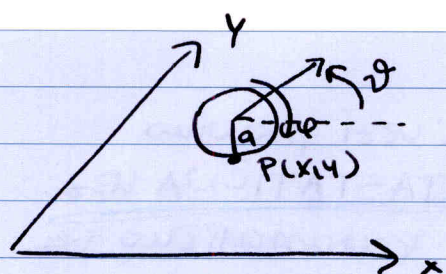
Primer 1: Kotanje diska po premici



Primer 2: Kotanje diska po ravlini:

P_0 = središče diska

P = dotikališče na ravlini



x, y, φ, ϑ + kotanjevje (vez)

$$\dot{\mathbf{P}}' = \dot{\mathbf{P}}_0' + \vec{\omega}' \times (\mathbf{P}' - \mathbf{P}_0') = \vec{0}$$

$$\dot{\mathbf{P}}_0' = \dot{x}\vec{i} + \dot{y}\vec{j} \quad ; \quad \vec{\omega}' = ?$$

$$\mathbf{P}' - \mathbf{P}_0' = -a\vec{k}$$

$$\mathbf{R} = \mathbf{R}(\vec{n}, \varphi) \circ \mathbf{R}(\vec{k}, \vartheta)$$

$$\vec{m} = \mathbf{R}(\vec{k}, \vartheta)\vec{i}$$

$$\Rightarrow \mathbf{R} = \mathbf{R}(\vec{k}, \vartheta) \circ (\vec{i}, \varphi)$$

$$\vec{\omega}' = \vartheta' \vec{k} + \varphi' \vec{m}$$

$$\vec{m} = \cos \vartheta \vec{i} + \sin \vartheta \vec{j}$$

$$\vec{\omega}' = \varphi' \cos \vartheta \vec{i} + \varphi' \sin \vartheta \vec{j} + \vartheta' \vec{k}$$

$$\vec{0} = (\dot{x}\vec{i} + \dot{y}\vec{j}) + (-a\varphi' \cos \vartheta)(-\vec{j}) + (-a\varphi' \sin \vartheta)\vec{i}$$

$$0 = \dot{x} + a\varphi' \sin \vartheta$$

$$0 = \dot{y} + a\varphi' \cos \vartheta$$

Azi je vez $0 = \dot{y} + a \cos \vartheta \dot{\varphi}$ integralna?

$$0 = 0 \cdot \dot{x} + 1 \cdot \dot{y} + a \cos \vartheta \dot{\varphi} + 0 \cdot \dot{\vartheta}$$

$$\frac{\partial(\lambda)}{\partial x} = 0 \quad \frac{\partial \lambda}{\partial \vartheta} = 0 \Rightarrow \lambda = \lambda(x, \varphi)$$

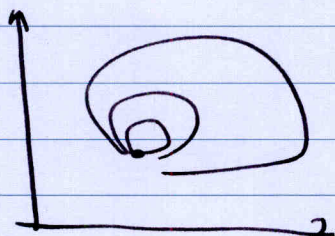
$$\frac{\partial \lambda}{\partial \varphi} = \frac{\partial (\lambda a \cos \vartheta)}{\partial \varphi}$$

$$\frac{\partial}{\partial x} (\lambda a \cos \vartheta) = 0$$

$$\frac{\partial}{\partial \vartheta} (\lambda a \cos \vartheta) = 0$$

$$-a\lambda \sin \vartheta = 0$$

tak $\lambda \neq 0$ me $\exists$!



Za λ iste x, y ϑ je φ lahko poizveden.

$\Downarrow$

mi vezi oblike $f(x, y, \vartheta, \varphi) = 0$

$$a_i(q, t) \dot{q}_i + a_0(q, t) = 0$$

$$a_0 = 0 \Rightarrow a_i(q, t) \dot{q}_i = 0$$

Taki vezi pravilno

KATASTATIČNA VEZ.

Skleronomna geometr. vez porabi katalastično vez.

Dejavski pomik (hitrost) - kakor & resnici

Možni pomik (hitrost)

Virtualni pomik (hitrost)

→ Pomik (hitrost), ki ustreza vsem vezem

Možna hitrost je rešitev sistema $\left[\begin{array}{l} a_i \cdot \dot{q}_i + a_{0i} = 0 \\ i = 1, \dots, m \end{array} \right]$

Virtualna hitrost pa je rešitev sistema

$$\left[\begin{array}{l} a_i \dot{q}_i = 0 \\ i = 1, \dots, m \end{array} \right]$$

Pravilno $q = (q_1, \dots, q_n) \in \mathbb{R}^n$

$$\left. \begin{array}{l} f_i(q, t) = 0 \quad i = 1, \dots, m' \\ a_{ij}(q, t) \dot{q}_j + a_{i0}(q, t) = 0 \\ i = 1, \dots, m'' \end{array} \right\} \left(\begin{array}{l} \frac{\partial f_i}{\partial q_j} \dot{q}_j + \frac{\partial f_i}{\partial t} = 0 \\ m' + m'' = m \text{ vezi} \end{array} \right)$$

Sistem vezi: $a_{ij}(q, t) \dot{q}_j + a_{i0}(q, t) = 0 \quad i = 1, \dots, m, \quad m < n$

Sistem vezi naj bo neodvisen $\Leftrightarrow \begin{array}{l} \text{def} \uparrow \\ \left[\frac{\partial a_{ij}}{\partial q_j} \right]_{\substack{i=1, \dots, m \\ j=1, \dots, n}} \text{ ima max. rang (=m)} \end{array}$

$n - m =:$ število prostostnih stopenj

Sistem vezi: $A \dot{q} + a_0 = 0$

$A \in \mathbb{R}^{m \times n}, \quad a_0 \in \mathbb{R}^m$

$A = A(q, t), \quad a_0 = a_0(q, t)$

Def. Možna hitrost je vsaka rešitev $\dot{q}$ sistema vezi

$$A\dot{q} + a_0 = 0.$$

Virtualna hitrost pa je vsaka rešitev $\delta\dot{q}$ sistema vezi $A\delta\dot{q} = 0$.

Očitno: če je $a_0 = 0$ se pojma virtualna in možna hitrost ~~sovpadata~~.
 $\Leftrightarrow$ sistem vezi je katastatičen

POSLEDICA Če je sistem holonom (vse vezi so geometrijske) in skleronomem (vse so neodvisne od časa) se pojma virtualna in možna hitrost ~~sovpadata~~ ^{upamata}.

Možni (infinitesimalni) pomik je $dq = \dot{q} dt$.

Virtualni (~~+~~) pomik je $\delta q = \delta\dot{q} dt$.

PRINCIP VIRTUALNEGA DELA

Klasifikacija sil $m\ddot{a} = \vec{F} = \vec{F}_m + \vec{F}_z$ ^{notr} ^{zun}
 $m\ddot{a} = \vec{F} = \vec{F}_1 + \vec{F}_2$
 rezultanta $\downarrow$ $\swarrow$ $\searrow$ rezult. pasivnih sil
 aktivnih sil

- Sila je PASIVNA, če je njeno virtualno delo $\vec{F} \cdot \delta\dot{q} = 0$ za vsak virtualni pomik.
- Sila $\vec{F}$ je AKTIVNA, če je $\vec{F} \cdot \delta\dot{q} \neq 0$ za vsaj en virt. pomik (= če ni pasivna)

Infinitesimalno delo sile $\vec{F}$ je $dA = \vec{F} \cdot \vec{v} dt = \vec{F} \cdot \dot{\vec{r}} dt$

Virtualno ~~delo~~ infinitesimalno delo sile $\vec{F}$ je $dA = \vec{F} \cdot \delta\dot{\vec{r}} dt = \vec{F} \cdot \delta\dot{\vec{r}} dt$.

aktivna: sila teže

PRIMER

pasivna sila je: ① pri horizontalnem premikanju
je sila teže pasivna

(nemika se $\perp$ na silo teže)

② Notranje sile pri togem sistemu

③ Pri enakomernem kroženju

④ Sila vazi pri gibanju po gladki ploskvi (krivulji)

⑤ Sila kotajenja

računski

PRIMER

Uvaj za sistem točk P_i velja $|P_i - P_j| = a_{ij}(t)$

Če so notranje sile centralne, je ^{skupno} virtualno delo

~~Rezultante~~ notr. sil enako nič.

$$|P_i - P_j|^2 = a_{ij}^2(t) \text{ je vez}$$

Vezi minimiramo sistem za virtualne hitrosti:

$$(P_i - P_j) \cdot (\dot{P}_i - \dot{P}_j) = a_{ij}(t) \dot{a}_{ij}(t)$$

$$(P_i - P_j) \cdot (\delta \dot{P}_i - \delta \dot{P}_j) = 0$$

$$\delta P_i = \delta \dot{P}_i dt$$

$\vec{F}_{ji} \cdot \delta P_i$... virtualno delo sile $\vec{F}_{ji}$ na P_i .

$$\text{skupno virt. delo: } \delta A = \sum_{i,j} \vec{F}_{ji} \delta P_i = \frac{1}{2} \sum_{i,j} \vec{F}_{ji} \delta P_i + \sum_{j,i} \vec{F}_{ij} \delta P_j =$$

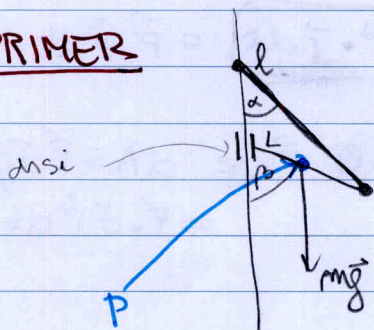
$$= \delta A = \frac{1}{2} \sum_{i,j} \vec{F}_{ji} (\delta P_i - \delta P_j) =$$

$$= \frac{1}{2} \sum \vec{F}_{ji} (|P_i - P_j|) \left(\frac{P_i - P_j}{|P_i - P_j|} \right) (\delta P_i - \delta P_j) = 0$$

Princip virtualnega dela:

Sistem je v ravnovesju $\Leftrightarrow$ virtualno delo aktivnih sil = 0
za $\forall$ virtualni pomik

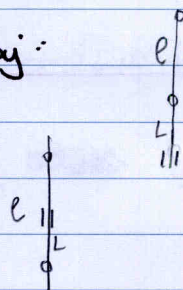
PRIMER



Poišči ravnovesni položaj:

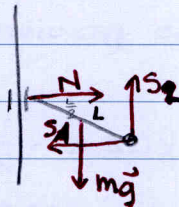
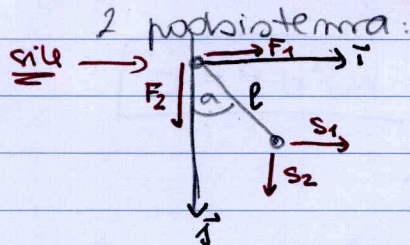
očitno 1) $\alpha = 0, \beta = \pi$

2) $\alpha = 0, \beta = 0$



3) ? Iskanje ~~izračun~~ oprnega ravnovesnega položaja

① REŠITEV Z VEKTORSKO MEHANIKO



minuje, ko vsota sil = 0 in vsota momentov = 0

$$\Rightarrow F_1 + S_1 = 0$$

$$F_2 + S_2 = 0$$

$$N - S_1 = 0$$

$$mg - S_2 = 0$$

$$\text{moment} - S_1 l \cdot \cos \alpha + S_2 l \sin \alpha = 0$$

$$(1) - S_2 L \sin \beta + S_1 L \cos \beta + mg \frac{L}{2} \sin \beta = 0$$

neznanke: $F_1, F_2, S_1, S_2, N, \alpha, \beta$ (med kotoma je zveza)

$$L \sin \beta = l \sin \alpha$$

$\leadsto$ 7 enačb, 7 neznanke

$$S_1 = \frac{\sin \alpha}{\cos \alpha} S_2 = \tan \alpha S_2 = mg \tan \alpha$$

$$\text{v (1): } -mg \sin \beta + mg \tan \alpha \cos \beta + mg \frac{1}{2} \sin \beta = 0 \quad | : \cos \beta$$

$$- \tan \beta + \tan \alpha + \frac{1}{2} \tan \beta = 0$$

$$\tan \alpha = \frac{1}{2} \tan \beta$$

$$\operatorname{ctg} \alpha = 2 \operatorname{ctg} \beta$$

$$1 + \operatorname{ctg}^2 \alpha = 4 \operatorname{ctg}^2 \beta + 1 = 4(1 + \operatorname{ctg}^2 \beta) - 3$$

$$\frac{1}{\sin^2 \alpha} = 4 \left(\frac{1}{\sin^2 \beta} \right) - 3 = 4 \frac{L^2}{l^2 \sin^2 \alpha} - 3$$

$$\frac{1}{\sin^2 \alpha} \left(1 - 4 \frac{L^2}{l^2} \right) = -3$$

$$\sin^2 \alpha = \frac{4 \frac{L^2}{l^2} - 1}{3} = \frac{4L^2 - l^2}{3l^2} > 0$$

$$\Rightarrow 4L^2 - l^2 > 0$$

$$4L^2 \geq l^2$$

$$\text{in: } \frac{4L^2 - l^2}{3} \leq 1$$

$$\boxed{L \leq l}$$

③

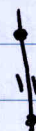
$$\boxed{L > \frac{1}{2}l}$$

Ta vmesna lega je nestabilna

$\frac{1}{2}l < L \leq l$ pri ②.

$\Rightarrow$ stabilna

Drugeče nestabilna.



② Reševanje s principom virt. dela

$$\delta A = 0$$

Sile F_1, F_2, S_1, S_2, N so pasivne

$$\boxed{m\vec{g} \cdot \delta P = 0}$$

$$P = O + \frac{L}{2} \sin \beta \vec{i} + (L \cos \alpha - \frac{L}{2} \cos \beta) \vec{j}$$

$$\delta P = \frac{\partial P}{\partial \alpha} \delta \alpha + \frac{\partial P}{\partial \beta} \delta \beta = \frac{L}{2} \cos \beta \delta \beta \vec{i} + (-L \sin \alpha \delta \alpha + \frac{L}{2} \sin \beta \delta \beta) \vec{j}$$

$$m\vec{g} \cdot \delta P = (mg \vec{j}) \cdot \delta P = (mg) \cdot (-L \sin \alpha \delta \alpha + \frac{L}{2} \sin \beta \delta \beta) = 0$$

$L \sin \beta = L \sin \alpha$: $L \cos \beta \cdot \delta \beta = L \cos \alpha \delta \alpha$

$\swarrow$ vst. $\delta \beta = \frac{L}{L} \frac{\cos \alpha}{\cos \beta} \delta \alpha$

$$(-L \sin \alpha \delta \alpha + \frac{L}{2} \frac{L}{L} \sin \beta \frac{\cos \alpha}{\cos \beta} \delta \alpha) = 0$$

$$(-\sin \alpha + \frac{1}{2} \cos \alpha \tan \beta) = 0$$

$$\tan \alpha = \frac{1}{2} \tan \beta$$

Zanimra nas še $\vec{N} = ?$

Virtuelna rot. okrog. O.

$$P \mapsto O + Q(\varphi) (P - O) = P(\varphi)$$

$$\delta P = \frac{\partial P}{\partial \varphi} \Big|_{\varphi=0} \delta \varphi$$

$$\delta P = \frac{dQ}{d\varphi} (P - O) \Big|_{\varphi=0} \delta \varphi$$

$$\dot{Q} = \frac{dQ}{dt} = \frac{dQ}{d\varphi} \cdot \dot{\varphi}$$

$$\delta P = \vec{k} \times Q(P - O) \Big|_{\varphi=0} \delta \varphi, \quad Q(\varphi=0) = I$$

$$\delta P = \vec{k} \times (P - O) \delta \varphi$$

$$\delta A = \vec{N} \cdot \delta P_1 + m\vec{g} \cdot \delta P = 0 \quad \vec{N} = N\vec{i}, \quad m\vec{g} = mg\vec{j}$$

$$P_1 - O = (L \cos \alpha - L \cos \beta) \vec{j}$$

$$\delta P_1 = \vec{k} \times (L \cos \alpha - L \cos \beta) \vec{j} \delta \varphi = -(L \cos \alpha - L \cos \beta) \vec{i} \delta \varphi$$

$$P - O = \frac{L}{2} \sin \beta \vec{i} + (L \cos \alpha - \frac{L}{2} \cos \beta) \vec{j}$$

$$\delta P = (\frac{L}{2} \sin \beta \vec{j} - (L \cos \alpha - \frac{L}{2} \cos \beta) \vec{i}) \delta \varphi$$

$$0 = N(-L \cos \alpha + L \cos \beta) \delta \varphi + mg \frac{L}{2} \sin \beta \delta \varphi \quad \checkmark \varphi$$

$$N = \frac{mg^{\frac{1}{2}} \sin \beta}{l \cos \alpha - L \cos \beta}$$

$$\delta A = \sum_{i=1}^N \vec{F}_i \cdot \delta P_i = 0$$

$$P_i = P_i(q) \quad , \quad q \in \mathbb{R}^m$$

$$\delta P_i = \sum \frac{\partial P_i}{\partial q_j} \delta q_j = \frac{\partial P_i}{\partial q_j} \delta q_j$$

$$\delta A = \underbrace{\left(\sum \vec{F}_i \cdot \frac{\partial P_i}{\partial q_j} \right)}_{Q_j} \delta q_j = Q_j \delta q_j$$

$\uparrow Q_j$ generalizirana sila

PRINCIP VIRTUALNEGA DELA

$$\delta A = Q_j \cdot \delta q_j = 0 \quad \forall q$$

- 1) koordinate q so neodvisne $\Rightarrow Q_j = 0 \quad \forall j$
(eno lahko spremenimo neodvisno od ostalih)
- 2) Koord. q so odvisne, $A \delta q = 0$, $A \in \mathbb{R}^{n \times m}$

$$Q \delta q = 0 \Leftrightarrow Q \perp \delta q$$

$$A \delta q = 0 \Leftrightarrow \delta q \in \ker(A) = (\operatorname{Im} A^T)^\perp$$

$$Q \in \operatorname{Im} A^T \Leftrightarrow Q_j = \sum_{k=1}^m \lambda_k A_{kj}$$

$$Q = A^T \lambda, \quad \lambda \in \mathbb{R}^m$$

$$\Rightarrow Q \cdot \delta q = 0$$

$$\checkmark A \cdot \delta q = 0$$

$$Q = A^T \lambda$$

+ m rezi

$\sqrt{n+m}$ $\leftarrow$ število enačb

$\begin{pmatrix} n \\ + \\ m \end{pmatrix} \rightarrow$ koord. ravnotež. polož. $\left\{ \begin{array}{l} \text{neznanke} \\ \lambda \in \mathbb{R}^m \end{array} \right.$

$$Q_j = \sum \vec{F}_i \cdot \frac{\partial P_i}{\partial q_j} = - \sum_{i=1}^N \left(\frac{\partial V_i}{\partial P_i} \right)^T \left(\frac{\partial P_i}{\partial q_j} \right) = - \sum_{i=1}^N \frac{\partial V_i}{\partial P_i} \cdot \frac{\partial P_i}{\partial q_j} = - \frac{\partial V}{\partial q_j}$$

$$F_i = - \left(\frac{\partial V_i}{\partial P_i} \right)^T = - \left(\frac{\partial V}{\partial P_i} \right)^T$$

$$\uparrow V(P_1, \dots, P_N) = V_1(P_1) + \dots + V_N(P_N)$$

$\checkmark$ so sile potencialne, $\&$ $Q_j = - \frac{\partial V}{\partial q_j}$ (so tudi gen. sile potencialne)

Primer (nadaljevanje)

$$\vec{F} = m\vec{g}, \quad V = -mgy = -mg(l \cos \alpha - \frac{l}{2} \cos \beta)$$

$$P = \frac{l}{2} \sin \beta \vec{i} + (l \cos \alpha - \frac{l}{2} \cos \beta) \vec{j}$$

$$0 = \frac{\partial V}{\partial \alpha} = -mg(-l \sin \alpha + \frac{l}{2} \sin \beta \frac{d\beta}{d\alpha}), \quad \beta = \beta(\alpha)$$

$$l \cdot \sin \alpha = L \sin \beta \Rightarrow l \cdot \cos \alpha = L \cos \beta \cdot \frac{d\beta}{d\alpha}$$

$$\Rightarrow 0 = -mg(-l \sin \alpha + \frac{l}{2} \sin \beta \cdot \frac{l \cdot \cos \alpha}{L \cos \beta}) \Rightarrow -l \sin \alpha + \frac{l}{2} \tan \beta \cos \alpha = 0$$

$$\Rightarrow \tan \alpha = \frac{1}{2} \tan \beta$$

Še rešitev z množitelji

$$\underline{Q} - A^T \underline{\lambda} = 0$$

$$\underline{Q} = (Q_1, Q_2), \quad Q_1 = -\frac{\partial V}{\partial \alpha} = -mgl \sin \alpha$$

$$Q_2 = -\frac{\partial V}{\partial \beta} = mg \frac{l}{2} \sin \beta$$

$$l \cdot \sin \alpha - L \sin \beta = 0$$

$$l \cdot \cos \alpha d\alpha - L \cos \beta d\beta = 0$$

$$a_{11} d\alpha + a_{12} d\beta = 0$$

$$a_{11} = l \cdot \cos \alpha$$

$$a_{12} = -L \cos \beta$$

$$A^T = \begin{bmatrix} l \cos \alpha \\ -L \cos \beta \end{bmatrix}$$

$$Q - \lambda A^T$$

$$Q_1 - \lambda a_{11} = 0$$

$$Q_2 - \lambda a_{12} = 0$$

$$-mgl \cdot \sin \alpha - \lambda l \cos \alpha = 0$$

$$mg \frac{l}{2} \sin \beta + \lambda L \cos \beta = 0$$

$$\lambda = -\frac{mg}{2} \tan \beta$$

in še enačba vezi: $l \sin \alpha - L \sin \beta = 0$

$$-mgl \sin \alpha + \tan \beta \frac{mg}{2} l \cos \alpha = 0$$

$$\Rightarrow \tan \alpha = \frac{1}{2} \tan \beta$$

VPRAŠANJE STABILNOSTI

Ravnovesna lega je stabilna, če ima v ravnovesni legi potencial svoj minimum.

$$V = -mg(l \cdot \cos \alpha - \frac{l}{2} \cos \beta)$$

$$0 = \frac{\partial V}{\partial \alpha} = -mg(-l \sin \alpha + \frac{l}{2} \sin \beta \cdot \frac{l \cos \alpha}{\cos \beta}) = mgl(\sin \alpha - \frac{1}{2} \cos \alpha \cdot \tan \beta)$$

$$\frac{d^2 V}{d\alpha^2} = mgl(\cos \alpha + \frac{1}{2} \sin \alpha \cdot \tan \beta - \frac{1}{2} \cos \alpha \cdot \frac{1}{\cos^2 \beta} \cdot \frac{l \cos \alpha}{\cos \beta} \cdot \frac{d\beta}{d\alpha})$$

$$\alpha = \beta = 0 \Rightarrow \frac{d^2 V}{d\alpha^2} = mgl \left(1 - \frac{1}{2} \frac{\rho}{L}\right) \text{ je stabilna} \Leftrightarrow 1 - \frac{1}{2} \frac{\rho}{L} > 0 \Leftrightarrow L > \frac{\rho}{2}$$

Lahko še izračunamo za mejni primer primer, ko je $L = \frac{\rho}{2}$
(izrač. višje odlode) DN

D' ALEMBERTOV PRINCIP

$$m\ddot{\mathbf{a}} = \vec{F}, \quad 0 = \vec{F} - m\ddot{\mathbf{a}} \quad | \delta \vec{r}$$

$$\sum_i (\vec{F}_i - m_i \ddot{\mathbf{a}}_i) \delta \vec{r}_i = 0$$

$$\sum \vec{F}_i \delta \vec{r}_i = Q_j \delta q_j = Q dq$$

$$\sum m_i \ddot{\mathbf{a}}_i \delta \vec{r}_i = \sum m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum m_i \left(\frac{d}{dt} \left(\dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} \right) - \dot{\vec{r}}_i \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} \right) \delta q_j$$

$$\vec{r}_i = \vec{r}_i(q, t)$$

$$\delta \vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$\frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial^2 \vec{r}_i}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} = \frac{\partial}{\partial q_j} \left(\frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

$$\dot{\vec{r}}_i = \dot{\vec{r}}_i(q, \dot{q}, t)$$

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k}$$

$$\dot{\vec{r}}_i \cdot \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} = \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \frac{d}{dq_j} \left(\frac{1}{2} \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \right)$$

$$\sum m_i \dot{\vec{r}}_i \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\frac{1}{2} \sum_{i=1}^N m_i |\dot{\vec{r}}_i|^2 \right) = \frac{\partial T}{\partial q_j}$$

$$\Rightarrow \dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial q_j} = \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\frac{1}{2} \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \right)$$

$$\Rightarrow \sum m_i \dot{\vec{r}}_i \delta \vec{r}_i = \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right) \delta q_j$$

$$\Rightarrow U = \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j \quad \text{SPOŠNA OBLIKA}$$

Najvelja $Q_j = - \frac{\partial V}{\partial q_j}$ $V = V(q, t)$ $\Rightarrow$

$$\Rightarrow 0 = \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} \right) \delta q_j$$

$$= \frac{d}{dt} \frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} \delta q_j$$

$$(*) \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right)^T \cdot \delta q = 0 \quad \forall \delta q$$

$$\Rightarrow \frac{\partial V}{\partial q_j} = 0$$

$$0 = \left(\frac{d}{dt} \frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} \right) \delta q_j$$

kin. energija T rot. energija V

$$\boxed{L = T - V} \rightarrow \text{Lagrangeva funkcija}$$

$$0 = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \right) \delta q_j$$

koordinate so neodvisne
(δq)

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0}, \quad j = 1, \dots, N$$

Lagrangeve enačbe

$$L = T - V$$

$$q(t=0) = q_0$$

$$\dot{q}(t=0) = \dot{q}_0$$

začetni pogoji

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q}$$

Velja tudi za togo telo, ne le za sistem materialnih točk.

$$L = L(q, \dot{q}, t)$$

celotna teorija je neodvisna od zač. enačbe

$L = T - V$ jo pa za dan problem tako zapišemo.

POBIVITEV

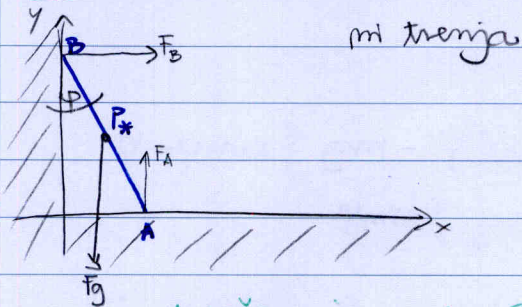
$$\delta q \text{ o.d.v.}, \quad A \delta q = 0$$

$$\delta q \in \ker A = (\text{Im } A^T)^\perp$$

$$\Rightarrow \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right)^T \in \text{Im } A^T$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = A^T \lambda, \quad \lambda \in \mathbb{R}^m$$

m enačb, m neznanke
+ m enačb
vezi q λ

PRIMER Zdrs palice ob gladki steni

1. reševanje preko Eulerjevih enačb — ZA MAKROSVET (kjer je trenja) ^{boljše}

$$m \vec{a}_* = \vec{F}' = m \vec{g} + \vec{F}_A + \vec{F}_B$$

koord. $P_x \left(\frac{l}{2} \sin \varphi, \frac{l}{2} \cos \varphi \right)$

$$\vec{v}_* = \frac{l}{2} \dot{\varphi} \cos \varphi \vec{i} - \frac{l}{2} \dot{\varphi} \sin \varphi \vec{j}$$

$$\vec{a}_* = \frac{l}{2} \ddot{\varphi} (\cos \varphi \vec{i} - \sin \varphi \vec{j}) + \frac{l}{2} \dot{\varphi}^2 (-\sin \varphi \vec{i} - \cos \varphi \vec{j})$$

$$m \vec{g} = -mg \vec{j}, \quad \vec{F}_A = F_A \vec{j}, \quad \vec{F}_B = F_B \vec{i}$$

$$\vec{i}: m \frac{l}{2} (\ddot{\varphi} \cos \varphi - \dot{\varphi}^2 \sin \varphi) = F_B$$

$$\vec{j}: -\frac{lm}{2} (\ddot{\varphi} \sin \varphi + \dot{\varphi}^2 \cos \varphi) = -mg + F_A$$

3 neznámke
2 enačbi.

IZREK O VRT. KOL:

$$J \ddot{\varphi} = \vec{N} \cdot \vec{r} =$$

$$J_* = \frac{1}{12} m l^2$$

$$\frac{1}{12} m l^2 \ddot{\varphi} = F_A \cdot \frac{l}{2} \sin \varphi - F_B \cdot \frac{l}{2} \cos \varphi$$

$$\frac{l}{2} F_A \sin \varphi - \frac{l}{2} F_B \cos \varphi = \frac{l}{2} mg \sin \varphi - \frac{m l^2}{4} (\ddot{\varphi} \sin \varphi + \dot{\varphi}^2 \cos \varphi) \sin \varphi - \frac{m l^2}{4} (\ddot{\varphi} \cos \varphi - \dot{\varphi}^2 \sin \varphi) \cos \varphi =$$

$$= \frac{l}{2} mg \sin \varphi - \frac{m l^2}{4} \ddot{\varphi}$$

$$\frac{1}{12} m l^2 \ddot{\varphi} = \frac{l}{2} mg \sin \varphi - \frac{m l^2}{4} \ddot{\varphi} \quad | \cdot 4$$

$$\frac{1}{3} l \ddot{\varphi} = \frac{1}{2} g \sin \varphi$$

$$\boxed{\ddot{\varphi} = \frac{3}{2} \frac{g}{l} \sin \varphi} \quad \text{reši!}$$

2. Lagrangeove enačbe, neodvisne koordinate — ZA MIKROSVET (določimo trenja)

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$$

$$L = T - V$$

kin. pot
energija

transl.

rot. energ

$$T = \frac{1}{2} m v_*^2 + \frac{1}{2} J \dot{\varphi}^2 = \frac{1}{2} m \frac{l^2}{4} \dot{\varphi}^2 + \frac{1}{2} \frac{1}{12} m l^2 \dot{\varphi}^2 = \frac{1}{2} m l^2 \frac{1}{3} \dot{\varphi}^2$$

$$V = mgy_* = mg \frac{\ell}{2} \cos \varphi$$

$$L = T - V = \frac{1}{2} m \ell^2 \frac{1}{3} \dot{\varphi}^2 - mg \frac{\ell}{2} \cos \varphi$$

$$\bullet \frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{3} m \ell^2 \dot{\varphi} \quad \leftarrow \text{to } \frac{d}{dt}$$

$$\bullet \frac{\partial L}{\partial \varphi} = mg \frac{\ell}{2} \sin \varphi$$

$$\bullet \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = \frac{1}{3} m \ell^2 \ddot{\varphi} - mg \frac{\ell}{2} \sin \varphi = 0$$

$$\Rightarrow \ddot{\varphi} = \frac{3}{2} \frac{g}{\ell} \sin \varphi$$

3. Lagrangian equations, admissible constraints.

$$x_*, y_*, \varphi$$

$$T = \frac{1}{2} m (\dot{x}_*^2 + \dot{y}_*^2) + \frac{1}{2} J \dot{\varphi}^2$$

$$V = mgy_*$$

$$y_* + x_* = y_1 x \quad \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right)^T = A^T \underline{\lambda}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{j=1}^m \lambda_j A_{ji}, \quad i=1, \dots, m$$

$$m=2 \quad (2 \text{ rezi})$$

$$x = \frac{\ell}{2} \sin \varphi, \quad y = \frac{\ell}{2} \cos \varphi \quad \Rightarrow \quad x - \frac{\ell}{2} \sin \varphi = 0, \quad y - \frac{\ell}{2} \cos \varphi = 0$$

$$\delta x = \frac{\ell}{2} \cos \varphi \delta \varphi = 0$$

$$\delta y = \frac{\ell}{2} \sin \varphi \delta \varphi = 0$$

} admissible

$$A \in \mathbb{R}^{2,3} = \begin{bmatrix} 1 & 0 & -\frac{\ell}{2} \cos \varphi \\ 0 & 1 & \frac{\ell}{2} \sin \varphi \end{bmatrix}$$

$$\frac{dL}{dt} = T + V$$

$$\frac{\partial L}{\partial x}$$

equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \lambda_1 \Rightarrow (m \ddot{x}) = \lambda_1 \Rightarrow$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda_2 \Rightarrow m \ddot{y} + mg = \lambda_2 \Rightarrow$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = -\lambda_1 \frac{\ell}{2} \cos \varphi + \lambda_2 \frac{\ell}{2} \sin \varphi \Rightarrow J \ddot{\varphi} = -\lambda_1 \frac{\ell}{2} \cos \varphi + \lambda_2 \frac{\ell}{2} \sin \varphi$$

$$\lambda_1 = m \left(\frac{\ell}{2} \ddot{\varphi} \cos \varphi \right) = \frac{m \ell}{2} (\ddot{\varphi} \cos \varphi - \dot{\varphi}^2 \sin \varphi)$$

$$\lambda_2 = -\frac{m \ell}{2} (\ddot{\varphi} \sin \varphi + \dot{\varphi}^2 \cos \varphi)$$

$$\lambda_3 = \dots$$

⋮

5 equations, 5 rezi

+ x, y, φ

(λ_1, λ_2)

koordinata $q, \underline{Q}$; $q = q(Q, t)$ difeomorfizem $\rightarrow \dot{q} = \frac{\partial q}{\partial Q} \dot{Q} + \frac{\partial q}{\partial t}$
 $\underline{Q} = Q(q, t)$

$$L = L(q, \dot{q}, t), \hat{L} = \hat{L}(Q, \dot{Q}, t) := L(q(Q, t), \frac{\partial q}{\partial Q} \dot{Q} + \frac{\partial q}{\partial t}, t)$$

$$\frac{d}{dt} \frac{\partial \hat{L}}{\partial \dot{Q}} + \frac{\partial \hat{L}}{\partial Q} = *$$

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} \cdot \frac{\partial \dot{q}}{\partial \dot{Q}}$$

$$\frac{d}{dt} \frac{\partial \hat{L}}{\partial \dot{Q}} = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \frac{\partial q}{\partial Q} + \frac{\partial L}{\partial \dot{q}} \left(\frac{\partial^2 q}{\partial Q^2} \dot{Q} + \frac{\partial^2 q}{\partial Q \partial t} \right)$$

$$\frac{\partial \hat{L}}{\partial Q} = \frac{\partial L}{\partial q} \frac{\partial q}{\partial Q} + \frac{\partial L}{\partial \dot{q}} \left(\frac{\partial^2 q}{\partial Q^2} \dot{Q} + \frac{\partial^2 q}{\partial Q \partial t} \right)$$

$$* = \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right] \frac{\partial q}{\partial Q} + \frac{\partial L}{\partial \dot{q}} \left(\frac{\partial^2 q}{\partial Q^2} \dot{Q} + \frac{\partial^2 q}{\partial Q \partial t} \right) - \frac{\partial^2 q}{\partial Q^2} \dot{Q} \frac{\partial L}{\partial \dot{q}} - \frac{\partial^2 q}{\partial Q \partial t} \frac{\partial L}{\partial \dot{q}}$$

$$\frac{d}{dt} \frac{\partial \hat{L}}{\partial \dot{Q}} + \frac{\partial \hat{L}}{\partial Q} = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right) \frac{\partial q}{\partial Q} \quad (*, *)$$

$\Rightarrow$ PREK Naj bo $q = q(Q, t)$ difeomorfizem. Potem je $(*, *)$.
 Pravimo, da so Lagrangeve enačbe kovariantne.
 (neodvisne od izbire generaliziranih koordinat)

vektor $\left(\frac{\partial L}{\partial \dot{q}} \right)^T =: p$ $\left. \begin{array}{l} \text{kanonično} \\ \text{konjugiran} \\ \text{moment} \end{array} \right\}$
 (generalizirani moment)

$$\dot{p} = \left(\frac{\partial L}{\partial q} \right)^T$$

Definicija Če je $\frac{\partial L}{\partial q_i} = 0$, pravimo, da je q_i ciklična spremenljivka

PREK Kanonično konj. moment ciklične spremenljivke je konstanta gibanja:

$$\frac{\partial L}{\partial q_i} = 0 \Rightarrow p_i \text{ konst. gibanja}$$

DOKAZ $\frac{\partial L}{\partial q_i} = 0 \Rightarrow \dot{p} = 0 \rightarrow p$ je konstanta

V Newth. konst. gibanja: energija

Definicija Jacobijeva energ. funkcija $\mathcal{E} = p \cdot \dot{q} - L(q, \dot{q}, t)$.

$\frac{d\mathcal{E}}{dt}$ odvod vzdolž rešitve Lagr. enačb

$$\text{" } \dot{p} \dot{q} + p \ddot{q} - \frac{\partial L}{\partial q} \dot{q} - \frac{\partial L}{\partial \dot{q}} \ddot{q} - \frac{\partial L}{\partial t} = \underbrace{\left(\dot{p} - \left(\frac{\partial L}{\partial \dot{q}} \right)^T \right) \dot{q}}_{=0} - \frac{\partial L}{\partial t}$$

$$\boxed{\frac{d\mathcal{E}}{dt} = -\frac{\partial L}{\partial t}} = 0, \text{ če } L \text{ neodv. od } t \text{ !!}$$

($\mathcal{E}$ konstanta)

Definicija Sistem je avtonomen, če je $\frac{\partial L}{\partial t} = 0$.

IZREK Če je sistem avtonomen, je Jacobijeva energ. f. konstanta gibanja.

$$\hat{L} = L + \frac{dF}{dt}, \quad F(q, t)$$

⋮

$$\frac{d}{dt} \frac{d\hat{L}}{dq} - \frac{\partial \hat{L}}{\partial q} = \frac{d}{dt} \frac{\partial L}{dq} - \frac{\partial L}{dq} \Rightarrow \text{Lagr. enačbe so invariantne za transformacijo}$$

$$L \mapsto L + \frac{dF}{dt}$$

PRIMER: Sistem mat. točk

$$T = \frac{1}{2} \sum_{i=1}^N m_i |\dot{P}_i|^2, \quad V = V(P_1, \dots, P_N)$$

$$P_i = P_i(q, t) \text{ (rezi)}, \quad q \in \mathbb{R}^m$$

$$L = T - V$$

$$\dot{P}_i = \frac{\partial P_i}{\partial q} \cdot \dot{q} + \frac{\partial P_i}{\partial t}$$

$$\dot{P}_i \cdot \dot{P}_i = \frac{\partial P_i}{\partial q} \dot{q} \frac{\partial P_i}{\partial q} \dot{q} + 2 \frac{\partial P_i}{\partial q} \dot{q} \frac{\partial P_i}{\partial t} + \frac{\partial P_i}{\partial t} \frac{\partial P_i}{\partial t}$$

$$T = \frac{1}{2} \sum m_i \left(\underbrace{\frac{\partial P_i}{\partial q} \dot{q} \frac{\partial P_i}{\partial q} \dot{q}} + \underbrace{2 \frac{\partial P_i}{\partial q} \dot{q} \frac{\partial P_i}{\partial t}} + \underbrace{\frac{\partial P_i}{\partial t} \frac{\partial P_i}{\partial t}} \right)$$

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \boldsymbol{\pi} \dot{\mathbf{q}} + \boldsymbol{\pi}_1 \dot{\mathbf{q}} + \pi_0 \quad \boldsymbol{\pi} \dots \text{sim. matrica } m \times m$$

$$L = T - V$$

$$\left(\frac{dL}{d\dot{\mathbf{q}}}\right)^T = \boldsymbol{\pi} \dot{\mathbf{q}} + \boldsymbol{\pi}_1 = \mathbf{p}$$

$$E = \dot{\mathbf{q}}^T \mathbf{p} - L = \dot{\mathbf{q}}^T \boldsymbol{\pi} \dot{\mathbf{q}} - \boldsymbol{\pi}_1 \cdot \dot{\mathbf{q}} - \frac{1}{2} \dot{\mathbf{q}}^T \boldsymbol{\pi} \dot{\mathbf{q}} - \boldsymbol{\pi}_1 \dot{\mathbf{q}} - \pi_0 + V = \frac{1}{2} \dot{\mathbf{q}}^T \boldsymbol{\pi} \dot{\mathbf{q}} + V - \pi_0$$

Eul. energ. funkcija

V splošnem $E \neq T + V$.

Posledica Če je sistem konzervativen in skleronomni, je $E = T + V$.

Primer Simetrična vrtavka

$$V = mg\ell \cos\vartheta$$

$$T = \frac{1}{2} \boldsymbol{\omega}^T \mathbf{J} \boldsymbol{\omega}$$

$$\boldsymbol{\omega} = (\dot{\varphi} \sin\vartheta \sin\psi +$$

$$\dot{\vartheta} \cos\psi) \vec{E}_1 + (\dot{\varphi} \sin\vartheta \cos\psi - \dot{\vartheta} \sin\psi) \vec{E}_2 + (\dot{\varphi} \cos\vartheta + \dot{\psi}) \vec{E}_3$$

$$\Rightarrow T = \frac{1}{2} (\omega_1^2 J_1 + \omega_2^2 J_2 + \omega_3^2 J_3) = \dots \text{vst. } \omega_1, \omega_2, \omega_3$$

$$L = T - mg\ell \cos\vartheta$$

$$\varphi, \psi \text{ sta ciklični spr.} \Rightarrow p_\varphi = \frac{dL}{d\dot{\varphi}} = J_1 \dot{\varphi} \sin\vartheta + J_3 (\dot{\varphi} \cos\vartheta + \dot{\psi}) \cos\vartheta$$

$$p_\psi = \frac{dL}{d\dot{\psi}} = J_3 (\dot{\varphi} \cos\vartheta + \dot{\psi})$$

konst. gibanja

↓ avtonomna ($\frac{dL}{dt} = 0$) $\Rightarrow$ E konst. gibanja

$$L = \frac{1}{2} \dot{\mathbf{q}}^T \boldsymbol{\pi} \dot{\mathbf{q}} - V$$

$$\mathbf{q} = \begin{bmatrix} \varphi \\ \vartheta \\ \psi \end{bmatrix} \quad \boldsymbol{\pi} = \begin{bmatrix} J_1 \sin^2\vartheta + J_3 \cos^2\vartheta & 0 & J_3 \cos\vartheta \\ 0 & J_1 & 0 \\ J_3 \cos\vartheta & 0 & J_3 \end{bmatrix}$$

$T + V$ je konstanta gibanja

IZREK (Noether) Naj bo dana Lagr. funkcija invariantna za družino difeomorfizmov $\mathbf{q} \mapsto \mathbf{h}_s(\mathbf{q})$, $s \in \mathbb{R}$ in naj

bo $\mathbf{h}_s$ diferenciable pri $s=0$. Potem je in $\mathbf{h}_{s=0} = \text{Id}$

$$I = \left(\frac{dL}{d\dot{\mathbf{q}}}\right)^T \left(\frac{d\mathbf{h}}{ds}\right)_{s=0} \text{ konstanta gibanja.}$$

Dokaz $L_s = L(\underline{h}_s(\underline{q}), \underbrace{\frac{d}{dt} \underline{h}_s(\underline{q})}_{\frac{d\underline{h}_s}{d\underline{q}} \dot{\underline{q}}}, t)$

$$L_s = L \quad \forall s$$

$$L(\underline{h}_s(\underline{q}), \frac{d\underline{h}_s}{d\underline{q}} \dot{\underline{q}}, t) = L(\underline{q}, \dot{\underline{q}}, t) \quad \forall s \quad \left(\cdot \frac{d}{ds} \right)_{s=0}$$

$$\frac{dL}{d\underline{q}} \frac{d\underline{h}}{ds}(\underline{q}) \Big|_{s=0} + \frac{dL}{d\dot{\underline{q}}} \frac{d}{ds} \left(\frac{d\underline{h}_s}{d\underline{q}} \right) \Big|_{s=0} \cdot \dot{\underline{q}} = 0$$

$$\frac{d}{dt} \frac{dL}{d\dot{\underline{q}}} - \frac{dL}{d\underline{q}} = 0$$

$$\frac{dL}{d\underline{q}} = \frac{d}{dt} \frac{dL}{d\dot{\underline{q}}} \quad ; \quad \frac{d}{dt} \left(\frac{dL}{d\dot{\underline{q}}} \right) \cdot \frac{d\underline{h}}{ds} \Big|_{s=0} + \frac{dL}{d\dot{\underline{q}}} \frac{d}{ds} \left(\frac{d}{dt} \underline{h}_s(\underline{q}) \right) \Big|_{s=0} = 0$$

adv. produkta

$$\Rightarrow \frac{d}{dt} \left(\left(\frac{dL}{d\dot{\underline{q}}} \right)^T \cdot \left(\frac{d\underline{h}}{ds} \right)_{s=0} \right) = 0 \quad \frac{d}{dt} \left(\frac{d}{ds} \underline{h}_s(\underline{q}) \right)$$

$$\Rightarrow p \left(\frac{d\underline{h}}{ds} \right)_{s=0} \text{ je nes konstanta gibanja}$$

primer 1 kraj to q_i ciklična koordinata.

$$\underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{-to mesto.}$$

zaradi cikličnosti

$$L(\underline{q}, \dot{\underline{q}}, t) = L(\underline{q} + s \underline{e}_i, \dot{\underline{q}}, t) \quad \forall s \quad (\text{ker koordin. ciklična})$$

$\underline{h}_s(\underline{q}) = \underline{q} + s \underline{e}_i$ L je invariantna za to druž. difeomorf.

$$\underline{I} = \left(\frac{dL}{d\dot{\underline{q}}} \right)^T \left(\frac{d\underline{h}}{ds} \right)_{s=0} = p \cdot \underline{e}_i = p_i$$

primer 2 $L = \frac{1}{2} \sum_{i=1}^n m_i |\dot{\underline{p}}_i|^2 - V(\underline{p}_1, \dots, \underline{p}_n)$ (= sistem makt. točk)

L je invariantna za rotacije okrog črte skozi $\underline{P}_0$.

$$\underline{P}_i \mapsto \underline{P}_0 + Q(s)(\underline{P}_i - \underline{P}_0) = \underline{P}'_i \quad Q(s=0) = \underline{I}$$

$$\underline{P} = (\underline{P}_1, \dots, \underline{P}_n)$$

$$\underline{P} \mapsto \underline{P}' = (\underline{P}'_1, \dots, \underline{P}'_n) = \underline{h}_s(\underline{P})$$

$$\underline{I} = \left(\frac{dL}{d\underline{p}} \right)^T \cdot \left(\frac{d\underline{h}_s}{ds} \right)_{s=0} \text{ je konstanta gibanja.}$$

$$\frac{d\underline{h}}{ds} \Big|_{s=0} = \left(\frac{d\underline{P}'_1}{ds} \Big|_{s=0}, \dots, \frac{d\underline{P}'_n}{ds} \Big|_{s=0} \right)$$

$$\frac{d\underline{P}'_i}{ds} \Big|_{s=0} = Q(s)(\underline{P}_i - \underline{P}_0) \Big|_{s=0} = Q(s)(\underline{\omega}(s) \times (\underline{P}_i - \underline{P}_0)) \Big|_{s=0} =$$

$$= \vec{e} \times (\vec{P}_i - \vec{P}_0)$$

$$\left(\frac{d\vec{h}}{ds}\right)_{s=0} = (\vec{e} \times (\vec{P}_1 - \vec{P}_0), \dots, \vec{e} \times (\vec{P}_n - \vec{P}_0))$$

$$\left(\frac{d\vec{L}}{ds}\right)^T = (m_1 \dot{\vec{P}}_1, \dots, m_n \dot{\vec{P}}_n)$$

skal. produkt: $I = \sum_{i=1}^n m_i \vec{P}_i \cdot (\vec{e} \times (\vec{P}_i - \vec{P}_0)) \stackrel{\text{ciklično zamenj vrstni red, } \vec{e} \text{ vn}}{=} \vec{e} \cdot \underbrace{\sum_{i=1}^n (\vec{P}_i - \vec{P}_0) \times m_i \vec{P}_i}_{\text{vrstna količina } \vec{L}(\vec{P}_0)}$

$$\boxed{I = \vec{e} \cdot \vec{L}(\vec{P}_0)}$$

Majhna mihomja okoli ravnovesne lege

V ima lokalni min. v q_0

Predpostavka q_0 stabilna $\Uparrow$ ravnovesna lega $\Leftrightarrow \left. \frac{\partial V}{\partial q} \right|_{q=q_0} = 0$,
 $\left. \frac{\partial^2 V}{\partial q^2} \right|_{q=q_0}$ je pozitivno definitna.

$$V \approx \frac{\partial^2 V}{\partial q^2} (q=q_0) \cdot \frac{1}{2} (q-q_0)^2$$

$$T = \frac{1}{2} \dot{q} \pi \dot{q}$$

$T+V=E$ konstanta gibanja

$L = T - V$ aproksimirano v okolici ravnovesne lege

$$O(q-q_0) = \delta$$

$$V = V(q_0) + \frac{\partial V}{\partial q}(q_0)(q-q_0) + \frac{1}{2} \frac{\partial^2 V}{\partial q^2}(q_0)(q-q_0)(q-q_0) \in O(\delta^3)$$

$$V \approx V(q_0) + \frac{1}{2} (q-q_0) \cdot V''(q-q_0)$$

$$T = \frac{1}{2} \dot{q} \pi \dot{q} = \frac{1}{2} \dot{q} \cdot (\pi(q) + \frac{\partial \pi}{\partial q}(q_0)(q-q_0) + O(\delta^2)) \cdot \dot{q} =$$

$$\left. \begin{array}{l} O(q-q_0) = \delta \\ T+V=E \end{array} \right\} \Rightarrow \dot{q} = O(\delta)$$

$$\boxed{L = \frac{1}{2} \dot{q} \pi \dot{q} - \frac{1}{2} (q-q_0) \cdot V''(q-q_0)} \quad (\text{apoks. Lagr. funkcija})$$

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \pi \ddot{q} + V''(q-q_0)$$

$$q \leftrightarrow q - q_0$$

$$\boxed{\pi \ddot{q} + V'' q = 0}$$

$$\mathbb{T} \ddot{\underline{q}} + \mathbb{V} \underline{q} = 0$$

$$\begin{aligned} \underline{q}(t) &= \overset{\text{konst}}{a} e^{i\omega t} \\ \dot{\underline{q}} &= i\omega a e^{i\omega t} \\ \ddot{\underline{q}} &= -\omega^2 e^{i\omega t} \cdot a = -\omega^2 \underline{q} \end{aligned}$$

$$\begin{aligned} \mathbb{T}(-\omega^2 \underline{q}) + \mathbb{V} \underline{q} &= 0 \\ (\mathbb{V} - \omega^2 \mathbb{T}) \underline{q} &= 0 \quad \omega = ? \end{aligned}$$

$$\boxed{\mathbb{V} \underline{q} = \omega^2 \mathbb{T} \underline{q}} \quad (\underline{A} \underline{x} = \lambda \underline{B} \underline{x})$$

$$\det(\mathbb{V} - \omega^2 \mathbb{T}) = 0 \quad (\text{poiščemo } \omega)$$

lahko tudi pot. det

posplošen problem
lastne vrednosti
simetrična

Trditev Pozitivni definitni $\mathbb{T}$ in $\mathbb{V}$ lahko literati dimenzijamo. $\mathbb{T}$ da obstaja matrika A , da je

$$\underline{A^T \mathbb{T} A = I} \quad \text{in} \quad \underline{A^T \mathbb{V} A = \Omega^2} \leftarrow \text{diagonalna matrika}$$

Dokaz V splošnem, za $\mathbb{T} \neq I$, $A \in O(3)$ (ni ortogonalna)

$$\mathbb{T} P = P \cdot \Lambda \quad \text{oz.} \quad P^T \mathbb{T} P = \Lambda$$

$$P \in O(3), \Lambda \text{ diag.}$$

$$\frac{1}{\sqrt{\Lambda}} P^T \mathbb{T} P \frac{1}{\sqrt{\Lambda}} = I \leftarrow \mathbb{T} \text{ pozit. def.}$$

$$(P \frac{1}{\sqrt{\Lambda}})^T \mathbb{T} (P \frac{1}{\sqrt{\Lambda}}) = I$$

$$(P \frac{1}{\sqrt{\Lambda}})^T \cdot \mathbb{V} \cdot (P \frac{1}{\sqrt{\Lambda}}) \cdot R = R \cdot \Omega^2, \quad R \in O(3)$$

$$(P \frac{1}{\sqrt{\Lambda}} R)^T \cdot \mathbb{V} (P \frac{1}{\sqrt{\Lambda}} R) = \Omega^2 \Rightarrow A = P \frac{1}{\sqrt{\Lambda}} R$$

Ω^2 diag. matr.

$$A^T \mathbb{T} A = \underbrace{R^T (\frac{1}{\sqrt{\Lambda}} P^T) T (P \frac{1}{\sqrt{\Lambda}}) R}_{=I} = I$$

$$A^T \mathbb{V} A = \Omega^2 \leftarrow A \text{ je nesing.}$$

$$A^T \mathbb{T} A = I$$

$$A^T \mathbb{V} A = I \Omega^2 = A^T \mathbb{T} A \Omega^2$$

$$A^T (\mathbb{V} A - \mathbb{T} A \Omega^2) = 0$$

$$\mathbb{V} A - \mathbb{T} A \Omega^2 = 0$$

$$\Rightarrow \boxed{\mathbb{V} A = \mathbb{T} A \Omega^2}$$

Rezitev pospe-sistema 1. vrsta. A je sest. iz l. vektorjev pospe.
problema lastne vrednosti

$$\mathbb{T} \ddot{\underline{q}} + \mathbb{V} \underline{q} = 0$$

$$\underline{q} = A \underline{\xi}$$

$$A^T / \quad \pi A \ddot{\xi} + V/A \xi = 0$$

$$\ddot{\xi} + \Omega^2 \xi = 0$$

$$\ddot{\xi}_i + \Omega^2 \xi_i = 0 \quad \xi_i = B_i \cos(\Omega t + \delta_i)$$

ξ normalizirane koordinate

RECEPT ① Rešimo posplošen problem lastnih vr., dobimo A in Ω^2

② Rešimo enačbo v norm.-koord., dobimo ξ

③ Rešitev je $q = A \xi$

$$T \ddot{q} + V q = 0$$

$$A^T T A = I$$

$$A^T V A = \Omega^2 \text{ diag. mat.}$$

TRDITEV Naj bosta T, V pozitivno definitni.

1. Vse l. vredn. so realne

2. so pozitivne

DOKAZ $V/a_i = \lambda_i T/a_i$

$$a_j V/a_i = a_j \lambda_i T/a_i \Rightarrow V/a_j a_i = a_j \lambda_j \cdot \lambda_i T/a_i$$

$$a \lambda_j b = \bar{\lambda} (a \cdot b) \quad \lambda_j T/a_j \cdot a_i$$

$$i=j \Rightarrow \lambda_i (\underbrace{T/a_i a_i}_{=1}) = \bar{\lambda}_i (a_i \cdot \underbrace{T/a_i}_{=1}) = \bar{\lambda} (T/a_i a_i) \Rightarrow \lambda_i = \bar{\lambda}$$

ξ normalizirane koordinate

$$\ddot{\xi} + \Omega^2 \xi = 0 \quad \text{(lagr. enačbe, vsaka norm. koord. imata s svojo lastno frekvenco)}$$

$$\ddot{\xi}_i + \omega_i^2 \xi_i = 0 \Rightarrow \xi_i = B_i \cos(\omega_i t + \delta_i)$$

$$\xi = \cos(\Omega t + \delta) B \quad \text{diag. matr}$$

$$q = A \cos(\Omega t + \delta) B$$

tač pogoji: $q(0) = q_0$

$$\dot{q}(t=0) = \dot{q}_0$$

$$q_0 = A \xi_0$$

$$\dot{q}_0 = A \dot{\xi}_0$$

$$A^T q_0 = A \cos \delta B$$

$$A^T \dot{q}_0 = \sin \delta B$$

$$A^T / \quad \dot{q}_0 = A \Omega \sin \delta B \quad (v t=0)$$

$$A^T \dot{q}_0 = \Omega \sin \delta B$$

$\rightarrow$ iz tega določimo $B, \delta = \dots$

primer: Dvojno mat. penalo

Privzemimo: $l_1 = l_2 = l$

$m_1 = m_2 = m$

koord: 2 kota

1. Ravnovesni položaj

$$\sim V = -mgl \cos \varphi_1 - mgl (\cos \varphi_1 + \cos \varphi_2)$$

$$\frac{\partial V}{\partial \varphi_1} = 2mgl \sin \varphi_1, \quad \frac{\partial V}{\partial \varphi_2} = mgl \sin \varphi_2$$

$$\varphi_1 = k_1 \pi$$

$$\varphi_2 = k_2 \pi$$

$$\frac{\partial^2 V}{\partial \varphi_1^2} = 2mgl \cos \varphi_1 \Big|_{\varphi_1 = k_1 \pi} = 2mgl (-1)^{k_1}$$

$$\frac{\partial^2 V}{\partial \varphi_2^2} = mgl \cos \varphi_2 \Big|_{\varphi_2 = k_2 \pi} = mgl (-1)^{k_2}$$

$$\Rightarrow V = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} mgl$$

$\varphi_1 = \varphi_2 = 0$ je stabilna ravnovesna lega

$$\sim T = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2$$

$$P_1 (l \sin \varphi_1, l \cos \varphi_1) \quad \vec{v}_1 = l \dot{\varphi}_1 (\cos \varphi_1 \vec{i} - \sin \varphi_1 \vec{j})$$

$$P_2 (l \sin \varphi_1 + l \sin \varphi_2, l \cos \varphi_1 + l \cos \varphi_2), \quad \vec{v}_2 =$$

$$\vec{v}_2 = l (\dot{\varphi}_1 \cos \varphi_1 + \dot{\varphi}_2 \cos \varphi_2) \vec{i} + (-\dot{\varphi}_1 \sin \varphi_1 - \dot{\varphi}_2 \sin \varphi_2) \vec{j}$$

$$v_1^2 = l^2 \dot{\varphi}_1^2$$

$$v_2^2 = l^2 (\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1 \dot{\varphi}_2 (\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2)) =$$

$$= l^2 (\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2))$$

$$\Rightarrow T = \frac{1}{2} m l^2 \dot{\varphi}_1^2 + \frac{1}{2} m l^2 \dot{\varphi}_2^2 + m l^2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$$

$$T = \frac{1}{2} m l^2 (2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2))$$

$$\frac{1}{2} m l^2 (2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1 \dot{\varphi}_2)$$

$$T = \frac{1}{2} \underline{q}^T \underline{\pi}(\underline{q}) \dot{\underline{q}} \rightarrow \frac{1}{2} \underline{q}^T \cdot \underline{\pi}(\underline{q}) \dot{\underline{q}}$$

$$\underline{\pi} = m l^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\underline{q}_0 \Rightarrow \varphi_1 = \varphi_2 = 0$$

režimo:

$$(V - \omega^2 T) \underline{q} = 0$$

$$mgl \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \omega^2 m l^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = mgl \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \omega^2 l \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right) =$$

$$\lambda = \omega^2 \frac{g}{c}, \quad \omega = \sqrt{\lambda \frac{g}{c}}$$

$$\begin{bmatrix} 2-\lambda & -\lambda \\ -\lambda & 1-\lambda \end{bmatrix} a = 0$$

$$g = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{vmatrix} 2(1-\lambda) & \lambda \\ \lambda & 1-\lambda \end{vmatrix} = 0$$

$$2(1-\lambda)^2 - \lambda^2 = 0$$

$$\Rightarrow \sqrt{2}(1-\lambda_1) - \lambda_1 = 0 \quad ; \quad \sqrt{2}(1-\lambda_2) + \lambda_2 = 0$$

$$\Rightarrow \lambda_1 = \frac{\sqrt{2}}{1+\sqrt{2}}$$

$$\lambda_2 = \frac{\sqrt{2}}{\sqrt{2}-1}$$

$$\frac{2+\sqrt{2}}{2+\sqrt{2}}$$

$$\frac{2-\sqrt{2}}{2-\sqrt{2}}$$

$$\omega_1 = \sqrt{2-\sqrt{2}} \sqrt{\frac{g}{c}}$$

$$\omega_2 = \sqrt{2+\sqrt{2}} \sqrt{\frac{g}{c}}$$

} splošna
rešitev je lin. kombinacija.

lastni vektorji $a = \begin{bmatrix} A \\ B \end{bmatrix}$

$$-2A + (1-\lambda)B = 0 \Rightarrow B = \frac{2A}{1-\lambda}$$

$$B_1 = \frac{(2-\sqrt{2})A_1}{\sqrt{2}-1} = \sqrt{2}A_1 \Rightarrow \underline{a}_1 = A_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

$$B_2 = \frac{(2+\sqrt{2})A_2}{-\sqrt{2}-1} = -\sqrt{2}A_2 \Rightarrow \underline{a}_2 = A_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

$$\underline{a}_1 \cdot \underline{a}_2 = 0 \quad \checkmark$$

$$\underline{a}_1 \cdot \underline{a}_2 = 1$$

$$1 = m \ell^2 A_1^2 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = m \ell^2 A_1^2 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \begin{bmatrix} 1+\sqrt{2} \\ 1+\sqrt{2} \end{bmatrix} = m \ell^2 A_1^2 (2+\sqrt{2}+\sqrt{2}+2) \Rightarrow A_1 = \frac{1}{\sqrt{m \ell^2 2 \cdot (2+\sqrt{2})}}$$

$$\underline{a}_2 \cdot \underline{a}_2 = 1$$

$$1 = m \ell^2 A_2^2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \Rightarrow A_2 = \frac{1}{\sqrt{m \ell^2 2(2-\sqrt{2})}}$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{m \ell^2 2(2+\sqrt{2})}} & \frac{1}{\sqrt{m \ell^2 2(2-\sqrt{2})}} \\ \frac{1}{\sqrt{m \ell^2 2+\sqrt{2}}} & \frac{-1}{\sqrt{m \ell^2 2-\sqrt{2}}} \end{bmatrix}$$

$$g = A \xi$$

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \leftarrow \text{mila } 2 \omega_1$$

$$\leftarrow \text{mila } 2 \omega_2$$

20.4.

Prvi osnovni način nihanja:

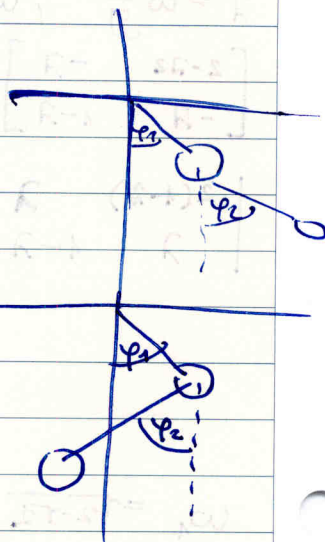
$$\varphi_1 = \frac{1}{\sqrt{2(2+\sqrt{2})}} \xi_1$$

$$\varphi_2 = \frac{1}{\sqrt{2+\sqrt{2}}} \xi_2$$

Drugi osnovni način nihanja:

$$\varphi_1 = \frac{1}{\sqrt{2(2-\sqrt{2})}} \xi_1$$

$$\varphi_2 = \frac{1}{\sqrt{2-\sqrt{2}}} \xi_2$$

DN: leži za 3 ali več~~POSPLOŠNI~~~~POTENCIJA~~

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q, \quad Q = - \frac{\partial V}{\partial q}, \quad (V = V(q))$$

$$\Rightarrow \frac{d}{dt} \frac{\partial (T-V)}{\partial \dot{q}} - \frac{\partial (T-V)}{\partial q} = 0, \quad L = T-V$$

Sila, ki je odvisna od hitrosti - Lorentzova sila

$$\vec{F} = q \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right)$$

 $\vec{E}$... električna poljska jakost $\vec{B}$... vektor magnetne indukcije q ... naboj, c hitrost svetlobe

$$Q = \frac{d}{dt} \frac{\partial V}{\partial \dot{q}} - \frac{\partial V}{\partial q}$$

rot $\vec{A} = ?$

$$\text{rot } \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Trditev $\frac{1}{2} \text{rot } \vec{A}$ je osni vektor ploskevne dela $\frac{\partial \vec{A}}{\partial P}$

$$\frac{1}{2} \left(\frac{\partial \vec{A}}{\partial P} - \left(\frac{\partial \vec{A}}{\partial P} \right)^T \right) \vec{a} = \frac{1}{2} \text{rot } \vec{A} \times \vec{a}$$

~~$$\left(\frac{dU}{d\vec{p}}\right)^T = -\frac{q}{c} \vec{A} \left(\frac{d}{dt} \left(\frac{dU}{d\vec{p}}\right)\right)^T = -\frac{q}{c} \frac{d\vec{A}}{dt}$$~~

$$\left(\frac{dU}{d\vec{p}}\right)^T = -\frac{q}{c} \vec{A} \quad ; \quad \left(\frac{d}{dt} \left(\frac{dU}{d\vec{p}}\right)\right)^T = -\frac{q}{c} \frac{d\vec{A}}{dt}$$

$$\left(\frac{dU}{d\vec{p}}\right)^T = q \left(\text{grad } \varphi - \frac{1}{c} \left(\frac{d}{dt} (\vec{A} \cdot \vec{p})\right)^T \right)$$

$$\left(\frac{d}{dt} \vec{u} \cdot \vec{v}\right)^T = \text{grad}(\vec{u} \cdot \vec{v}) = (\text{grad } \vec{u})^T \vec{v} + (\text{grad } \vec{v})^T \vec{u}$$

$$(\vec{u} \cdot \vec{v})(\vec{p} + \vec{q}) - (\vec{u} \cdot \vec{v})(\vec{p}) - \vec{A} \cdot \vec{h} = \sigma(\vec{u})$$

$$\vec{v} = \vec{a} = \text{konst.}$$

$$\text{grad}(\vec{u} \cdot \vec{a}) = (\text{grad } \vec{u})^T \vec{a}$$

manjka dokaz

$$\Rightarrow \left(\frac{dU}{d\vec{p}}\right)^T = q \left(\text{grad } \varphi - \frac{1}{c} \left(\frac{d}{dt} (\vec{A} \cdot \vec{p})\right)^T \right) = q \left(\text{grad } \varphi - \frac{1}{c} \text{grad}(\vec{A} \cdot \vec{p}) \right) =$$

$$= q \left(\text{grad } \varphi - \frac{1}{c} \left(\frac{d\vec{A}}{dt}\right)^T \cdot \vec{p} \right)$$

$$\Rightarrow \left(\frac{d}{dt} \frac{dU}{d\vec{p}} - \frac{dU}{d\vec{p}}\right)^T = -\frac{q}{c} \frac{d\vec{A}}{dt} + \frac{1}{c} \left(\frac{d\vec{A}}{dt}\right)^T \vec{p} = \vec{F}$$

Trditev Posplošeni potencial Lorentzove sile

$$\vec{F} = q(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}) \quad \& \quad U = q\left(\varphi - \frac{1}{c} \vec{A} \cdot \vec{v}\right)$$

$$L = T - U$$

$$U = q\left(\varphi - \frac{1}{c} \vec{A} \cdot \vec{v}\right)$$

$$\Lambda = \Lambda(\vec{p}, t)$$

$$\varphi \mapsto \varphi + \frac{d\Lambda}{dt}$$

$$\frac{d\Lambda}{dt} = \frac{\partial \Lambda}{\partial t} + \text{grad } \Lambda \cdot \vec{v}$$

$$\vec{A} \mapsto \vec{A} - c \text{ grad } \Lambda$$

$$U \mapsto U + q \left(\frac{\partial \Lambda}{\partial t} + \text{grad } \Lambda \cdot \vec{v} \right) = U + q \frac{d\Lambda}{dt}$$

$$L \mapsto L - q \frac{d\Lambda}{dt} \quad \text{ekvivalentna Lagrangena funkcija}$$



gibalne enačbe so invariantne za transformacijo

$$\varphi \mapsto \varphi + \frac{d\Lambda}{dt}, \quad \vec{A} \mapsto \vec{A} - c \text{ grad } \Lambda$$

(umeritvena transformacija)

Zapis Lagrangeove funkcije za rel. gibanje

$$L = \frac{1}{2} \sum m_i |\dot{\mathbf{r}}_i|^2 - U(\mathbf{r})$$

$$P_i' = P_0' + Q(P_i - P_0)$$

$$\dot{P}_i' = \dot{P}_0' + \bar{\omega} \times (P_i' - P_0') + Q \dot{P}_i$$

$\dot{P}_0^I = 0$ privzamenno (mi troušeatorne cutroosts)

$$|\dot{\mathbf{P}}_i|^2 = |\bar{\omega} \times (\mathbf{P}_i - \mathbf{P}_0)|^2 + |Q \dot{\mathbf{P}}_i|^2 + 2Q \dot{\mathbf{P}}_i \cdot (\bar{\omega} \times (\mathbf{P}_i - \mathbf{P}_0)) -$$

$$= |\vec{\omega} \times (\vec{P}_i - \vec{P}_0)|^2 + |\dot{\vec{P}}_i|^2 + 2 \dot{\vec{P}}_i \cdot (\vec{\omega} \times (\vec{P}_i - \vec{P}_0))$$

$$L = \frac{1}{2} \sum m_i |\dot{\mathbf{P}}_i|^2 + \left(\frac{1}{2} \sum m_i |\vec{\omega} \times (\mathbf{P}_i - \mathbf{P}_0)|^2 + \sum m_i \dot{\mathbf{P}}_i \cdot (\vec{\omega} \times (\mathbf{P}_i - \mathbf{P}_0)) - U(\mathbf{P}_0 + \mathbf{Q}(\mathbf{P} - \mathbf{P}_0)) \right)$$

$$'' - V_{rel}(\underline{P}, \underline{\dot{P}})$$

$$\Rightarrow L = \frac{1}{2} \sum m_i |\dot{\mathbf{p}}_i|^2 - V_{\text{rel}}(\underline{\mathbf{p}}, \underline{\dot{\mathbf{p}}})$$

↑ prospošeni potencial

$$V_p = -\frac{1}{2} \sum m_i |\omega \times (P_i - P_0)|^2 - \sum m_i \dot{P}_i \cdot (\omega \times (P_i - P_0)) \quad | \quad j^e$$

posplošeni potencial za vrsto centrifugalne,
Gardolisa in imerzijske vrste interakcije = $\left(\frac{d}{dV} \right) dV_p \quad dV_p / r$

Coriolisowe im inercijske sile rotacije = $\left(\frac{d}{dt} \frac{dV_p}{ds} - \frac{dV_p}{ds} \right) r$

manjka

TRDITEV Vektorski potencial konstantnega magnetnega polja $\vec{B}$ je $\vec{A} = \frac{1}{2} \vec{B} \times (\vec{r} - \vec{r}_0)$

Dokaz $\operatorname{rot} \vec{A} = \vec{B}$

Dokaz $\frac{d}{dt} \vec{A} = \vec{B}$
 $\text{rot } \vec{A} = \vec{W} \left(\frac{\partial \vec{A}}{\partial p} - \left(\frac{\partial \vec{A}}{\partial p} \right)^T \right)$

$$\frac{\partial A}{\partial \vec{p}} = \frac{\partial}{\partial \vec{p}} \left(\frac{1}{2} \vec{B} \times (\vec{p} - \vec{p}_0) \right) = \frac{1}{2} W(\vec{B})$$

$$\left(\frac{\partial \lambda}{\partial \vec{p}}\right)^T = \left(\frac{1}{2} W(\vec{B})\right)^T \stackrel{\uparrow \text{od meji}}{=} \frac{1}{2} W(-\vec{B}) = -\frac{1}{2} W(\vec{B})$$

$$\Rightarrow \frac{\partial A}{\partial p} - \left(\frac{\partial A}{\partial p} \right)^T = \frac{1}{2} W(\vec{B}) + \frac{1}{2} W(\vec{B}) = W(\vec{B})$$

$$\rightarrow W \left(\frac{\partial A}{\partial p} - \left(\frac{\partial A}{\partial p} \right)^T \right) = \vec{B} \quad \square$$

PRIMER opis gibanja v RKS regija v konst. magn. polju.
 $\therefore$ manjka

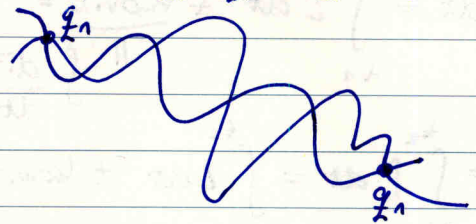
Hamiltonov variacijski princip

DEF. Aksijski funkcional $q = q(t) \mapsto I(q) = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \in \mathbb{R}$
 $\underline{q \mapsto I(q)}$

Dopustni prostor trajektorij =

mišljene
predpisane
vrednosti

$q_0 + D_0 = D = \{ q : q \in C^2([t_1, t_2]) \text{ in } q(t_1) = q_1, q(t_2) = q_2 \}$
 Očito je dopustni prostor trajektorij ~~afini~~ afini prostor.



Ekstremalne vrednosti

$q_0 \in D, \eta \in D_0$ ($q(t_1) = \eta(t_2) = 0$)

$q = q_0 + \alpha \eta \in D$

$\alpha \mapsto I(q_0 + \alpha \eta) = J(\alpha)$

$\mathbb{R} \rightarrow \mathbb{R}$

kaj ima $J(\alpha)$ ekstrem?

$$\begin{aligned} \left. \frac{dJ}{d\alpha} \right|_{\alpha=0} &= \left. \frac{d}{d\alpha} \int_{t_1}^{t_2} L(q_0 + \alpha \eta, \dot{q}_0 + \alpha \dot{\eta}, t) dt \right|_{\alpha=0} = \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}(q, \dot{q}, t) \eta + \frac{\partial L}{\partial \dot{q}}(q, \dot{q}, t) \dot{\eta} \right) dt = \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \eta dt + \left. \frac{\partial L}{\partial \dot{q}} \eta \right|_{t_1}^{t_2} = 0 \end{aligned}$$

$$\left. \frac{dJ}{d\alpha} \right|_{\alpha=0} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \eta dt = \left. \frac{\partial I}{\partial q} \right|_{q=q_0} [\eta]$$

Kaj je $\left. \frac{dJ}{d\alpha} \right|_{\alpha=0} = 0$ za $\forall \eta \in D_0$? $\Leftrightarrow \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \eta dt = 0 \quad \forall \eta \in D_0$

V vomiacijskem računu pravimo (tež) enačbi Euler-Lagrangea.

Po osnovni Cerni var. računa $\Leftrightarrow \frac{dL}{dq} - \frac{d}{dt} \frac{dL}{dq} = 0$.

Akcijska ima stacionarno vrednost $\checkmark$ ^{kritični} Lagrangeve enačb. (= g. je rešitev Lagr. enačb)

$$\hat{L} = L + \frac{dF}{dt}(q, t)$$

$$\int_{t_1}^{t_2} \hat{L} dt = \int_{t_1}^{t_2} L dt + \text{konst} =$$

$$\int_{t_1}^{t_2} \frac{dF}{dt}(q, t) dt = F(q_2, t_2) - F(q_1, t_1).$$

$$\hat{I}(q) = \int_{t_1}^{t_2} \hat{L} dt = \int_{t_1}^{t_2} L dt + \text{konst}$$

I in $\hat{I}$ imata iste stacionarne trajektorije.

Konec Lagrangeve mehanike.

3. HAMILTONOVA mehanika

Legendrova transformacija

$$y = f(x)$$

$$\frac{df}{dx} = z \Rightarrow x = x(z)$$

$$\frac{d}{dz} f(x(z)) = z$$

$$(\mathcal{L}f)(z) = z \cdot x(z) - f(x(z))$$

primer

$$f(x) = \frac{1}{\alpha} x^\alpha, \alpha \neq 1$$

$$\frac{df}{dx} = x^{\alpha-1} = z \Rightarrow x = z^{\frac{1}{\alpha-1}}$$

$$(\mathcal{L}f)(z) = z \cdot z^{\frac{1}{\alpha-1}} - \frac{1}{\alpha} (z^{\frac{1}{\alpha-1}})^\alpha = z^{1+\frac{1}{\alpha-1}} - \frac{1}{\alpha} z^{\frac{\alpha}{\alpha-1}} = \left(1 - \frac{1}{\alpha}\right) z^{\frac{\alpha}{\alpha-1}}$$

$$(\mathcal{L}f)(z) = \beta \cdot z^\beta; \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

V tem primeru $\mathcal{L}(\mathcal{L}f) = \frac{1}{2} x^\alpha$ (oz. Id)

inv. npr. transpon.

Trditelj $\mathcal{L}(\mathcal{L}) = \text{Id}$ Legendrova transf. je ~~involutivna~~ involutivna.

Dokaz $(\mathcal{L}f)(z) = z \cdot x(z) - f(x(z)) =: g(z)$

$$\frac{df}{dx}(x(z)) = z$$

$$(\mathcal{L}g)(u) = u \cdot z(u) - g(z(u)) = *$$

$$\frac{dg}{dz} = u \Rightarrow \frac{dg}{dz}(z(u)) = u$$

$$u = \frac{dg}{dz} = \frac{d}{dz}(z \cdot x(z) - f(x(z))) = x(z) + z \cdot \frac{dx}{dz} - \underbrace{\frac{df}{dx}(x(z)) \cdot \frac{dx}{dz}}_{=z} \quad u = x(z)$$

$$u = x(z) \Rightarrow z = z(u)$$

u, kje x iz inverzni

$$* = u(z(u)) - z(u) \cdot x(z(u)) + f(x(z(u))) = f(u)$$

$$\Rightarrow \underline{\underline{\mathcal{L}(\mathcal{L}f) = f}}$$

Def: Hamiltonova funkcija je Legendrova transformirana ~~te~~ Lagrangeove funkcije po spr. $\dot{q}$.

$$L = L(q, \dot{q}, t)$$

$$p = \left(\frac{\partial L}{\partial \dot{q}} \right)^T \Rightarrow \dot{q} = \dot{q}(q, p, t) \quad ; \quad \left(\frac{\partial^2 L}{\partial \dot{q}^2} \text{ neregularno} \right)$$

$$\underline{H = p \cdot \dot{q} - L(q, \dot{q}, t)} \quad \text{Hamiltonova funkcija}$$

Jacobijeva energ. funkcija

$$\underline{\tilde{E} = p \cdot \dot{q} - L(q, \dot{q}, t)}$$

na tang. prostoru $\tilde{E} = \text{funkcija } \tilde{q} \rightarrow \dot{q}, p = \left(\frac{\partial L}{\partial \dot{q}} \right)^T$

$$H = H(q, p, t)$$

H definirana na kotangentni svesnjici na faze prostoru.

Hamiltonova in Lagr. meh. sta ekvivalentni.

$$\begin{aligned} \left(\frac{\partial H}{\partial p} \right)^T &= \left(\frac{\partial}{\partial p} (p \cdot \dot{q} - L) \right)^T = \dot{q} + \left(\frac{\partial \dot{q}}{\partial p} \right)^T \cdot p - \left(\frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial p} \right)^T = \dot{q} + \left(\frac{\partial \dot{q}}{\partial p} \right)^T \cdot p - \\ &- \underbrace{\left(\frac{\partial \dot{q}}{\partial p} \right)^T \left(\frac{\partial L}{\partial \dot{q}} \right)^T}_{=p} = \dot{q}. \end{aligned}$$

$$\left(\frac{\partial H}{\partial \dot{q}} \right)^T = \left(\frac{d}{dt} (p \dot{q} - L) \right)^T = \left(\frac{\partial \dot{q}}{\partial \dot{q}} \right)^T p - \left(\frac{\partial L}{\partial \dot{q}} \right)^T - \left(\frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial \dot{q}} \right)^T =$$

$$= \dots = - \left(\frac{\partial L}{\partial \dot{q}} \right)^T.$$

Truditev $\left(\frac{dH}{dp} \right)^T = \dot{q}$ ← ta sklop enačb je ekvivalenten def. Legendrovi transformaciji

$$\left(\frac{\partial H}{\partial q} \right)^T = - \left(\frac{\partial L}{\partial q} \right)^T$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$$\dot{p} = \left(\frac{\partial L}{\partial q} \right)^T.$$

izrek q je rešitev Lagr. enačb $\Leftrightarrow$ par (q, p) je rešitev kanonskega sistema

$$\begin{cases} \dot{q} = \left(\frac{\partial H}{\partial p} \right)^T \\ \dot{p} = - \left(\frac{\partial H}{\partial q} \right)^T \end{cases} \leftarrow \text{rešujemo ta sistem}$$

↳ lažje rešujemo (numerčno)

PRIMER 1 $L = T - V = \frac{1}{2} q^T \pi \dot{q} - V(q, t)$ Poišči H. funkcijo.

$$p = \left(\frac{\partial L}{\partial \dot{q}} \right)^T = \pi \dot{q} \Rightarrow \dot{q} = \pi^{-1} p$$

$$H = \dot{q} p - L = \pi^{-1} p \cdot p - \frac{1}{2} (\pi^{-1} p)^T \cdot \pi (\pi^{-1} p) + V =$$

$$= \pi^{-1} p \cdot p - \frac{1}{2} \pi^{-1} p \cdot p + V = \underline{\underline{\frac{1}{2} p^T \pi^{-1} p + V}} \leftarrow \text{Ham. funkcija}$$

PRIMER 2 Gibanje v polju centralne sile

$$L = T - V = \frac{1}{2} m (\ddot{r} + (r\dot{\varphi})^2) - V(r)$$

$$\pi = m \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \quad | \quad \pi^{-1} = \frac{1}{m} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}$$

$$H = \frac{1}{2m} \begin{bmatrix} p_r \\ p_\varphi \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix} \begin{bmatrix} p_r \\ p_\varphi \end{bmatrix} + V(r) = \underline{\underline{\frac{1}{2m} \left(p_r^2 + \frac{1}{r^2} p_\varphi^2 \right) + V(r)}} \leftarrow \text{Ham. funkcija}$$

kanonski sistem: $\dot{q} = \left(\frac{\partial H}{\partial p} \right)^T$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{1}{m} p_r$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{1}{m} \frac{1}{r^2} p_\varphi$$