

1. NEWTONOVA MEHANIKA

Osnova Newtonove mehanike

Def

Afimi prostor $\mathcal{A}$ med velet. prostorom V je množica skupaj z operacijo $\mathcal{A} \times V \rightarrow \mathcal{A}$, za katero velja:

$$(A, \vec{a}) \mapsto A + \vec{a}$$

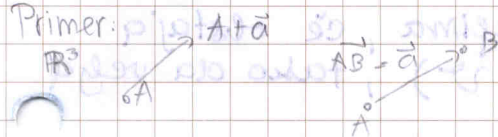
$$1. (A + \vec{a}) + \vec{b} = A + (\vec{a} + \vec{b})$$

2. za poljubna $A, B \in \mathcal{A} \exists$ natanko določen vektor $\vec{a} \in V$ da je $B = A + \vec{a}$

Elementom $\mathcal{A}$ ja pravimo dogodki.

Primer:

Npr. Evklidski prostor



Def

Nad $\mathcal{A} \times \mathcal{A} \rightarrow V$ je definirana operacija, ki $(A, B) \mapsto \vec{a}$, tako da je $B = A + \vec{a}$

$$\text{Pisava: } \vec{a} = B - A \Leftrightarrow B = A + \vec{a}$$

$$\bullet A + \vec{a} = A + \vec{b} \Leftrightarrow \vec{a} = \vec{b}$$

$B = A + \vec{a}$ in $B = A + \vec{b}$
iz 1. def. $\vec{a} = \vec{b}$ ($\exists$ ujed. en vektor)

Trditve

Velja:

$$1. A - A = \vec{0}$$

$$2. (A - B) + (B - A) = \vec{0}$$

$$3. (A - B) + (B - C) + (C - A) = \vec{0}$$

$$4. (A + \vec{a}) - B = (A - B) + \vec{a}$$

Dokaz

$$1. A - A = \vec{a} \xLeftrightarrow{\text{def}} A = A + \vec{a}$$

$$A + (-\vec{a}) = (A + \vec{a}) + (-\vec{a}) = A + (\vec{a} - \vec{a}) = A + \vec{0} \Rightarrow -\vec{a} = \vec{0}$$

$$2. \left. \begin{array}{l} A - B = \vec{a} \xLeftrightarrow{\text{def}} A = B + \vec{a} \\ B - A = \vec{b} \xLeftrightarrow{\text{def}} B = A + \vec{b} \end{array} \right\} \Rightarrow B = B + \vec{a} + \vec{b} \Rightarrow \vec{a} + \vec{b} = \vec{0}$$

Def

Galilejeva struktura je trojica $(\mathcal{A}, t, p)$, kjer je

1. $\mathcal{A}$ štirirazsežni afini prostor nad V
2. $t \in \mathcal{L}(V, \mathbb{R}) \leftarrow$ linearna operacija iz V v $\mathbb{R}$
3. p je evklidska metrika na $V_0 = \text{Ker } t$. Jedno +.

t je pretečeni čas med dvema dogodkoma.

Dogodka A, B , $t(B - A) = 0$ takima dogodkoma pravimo istovrstna dogodka.

Alkta

Metrika je definirana na prostoru istovremeni dogodkov.

Def Galilejevi strukturi

$$g = (A, t, s) \quad , \quad \hat{g} = (\hat{A}, \hat{t}, \hat{s})$$

sta ekvivalentni, če obstaja bijekcija $g: \mathcal{A} \rightarrow \hat{\mathcal{A}}$, da velja:

- 1 - g afina preslikava
- 2 - g ohranja časovnost
- 3 - g ohranja razdaljo med istovremeni dogodki

Preslikava g ima ime: Galilejeva transformacija.

Def Afina preslikava: $g: \hat{\mathcal{A}} \rightarrow \mathcal{A}$ je afina, če obstaja linearna preslikava $S \in \mathcal{L}(\vec{V}, \vec{V})$, tako da velja

$$g(\hat{A}) - g(\hat{B}) = S(\hat{A} - \hat{B}).$$

$$g(\hat{A}) = g(\hat{B}) + S(\hat{A} - \hat{B})$$

Afina preslikava je dana z linearno preslikavo S in vrednostjo preslikave τ pojavnem dogodku.

$$g: S, g(0) \Rightarrow g(A) = g(0) + S(A - 0), \text{ dogodku } 0 \text{ pravimo pol preslikave}$$

če O_1 :

$$g: S, g(O_1) \Rightarrow g(A) = g(O_1) + S(A - O_1)$$

$$\begin{aligned} O_1 &= 0 + (O_1 - 0) \\ g(O_1) &= g(0) + S(O_1 - 0) \end{aligned}$$

$$\Rightarrow g(A) = g(0) + S((A - O_1) + (O_1 - 0))$$

$$= g(0) + S(A - 0) + S(O_1 - 0)$$

Naravna Galilejeva struktura na $\mathbb{R} \times \mathbb{E}$

$$(t, P) \in \mathbb{R} \times \mathbb{E}$$

razd. med mestoi dogodki

↑ trirazsežni evklidski prostor

$$\mathbb{R} \times \mathbb{E} \text{ afini prostor: } \checkmark$$

$$(t, P) + (t_0, \vec{a}) \mapsto (t + t_0, P + \vec{a})$$

$$A_1 = (t_1, P_1)$$

$$A_2 = (t_2, P_2)$$

$$t(A_2 - A_1) = t_2 - t_1$$

$$A_2 \text{ in } A_1 \text{ sta istovremena} \Leftrightarrow t_1 = t_2$$

$$t_1 = t_2; \quad g(A_1, A_2) = \|P_1 - P_2\|$$

Kdaj sta dve Galilejevi strukturi na $\mathbb{R} \times \mathbb{E}$ ekvivalentni?

$$\left. \begin{aligned} a &\in \mathcal{O}(3), \alpha = 1 \\ b &= 0 \end{aligned} \right\}$$

$$g: \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{R} \times \mathbb{E} \quad \text{afina (bijekcija)}$$

$$(t, P) \mapsto (t', P')$$

$$g(t, P) = g(t_0, P_0) + S((t, P) - (t_0, P_0))$$

$$S \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^4)$$

$$S = \begin{bmatrix} \alpha & b \\ c & Q \end{bmatrix}$$

$$\begin{aligned} Q &\in \mathbb{R}^{3 \times 3} \\ \alpha &\in \mathbb{R} \\ c &\in \mathbb{R}^{3 \times 1} \\ b &\in \mathbb{R}^{1 \times 3} \end{aligned}$$

je obrambiva (bijelice) ✓

$$\begin{aligned} g(t, P) &= (t', P') \\ g(t_0, P_0) &= (t_0', P_0') \end{aligned}$$

$$\begin{bmatrix} t' \\ P' \end{bmatrix} = \begin{bmatrix} t_0' \\ P_0' \end{bmatrix} + \begin{bmatrix} a & b \\ c & Q \end{bmatrix} \begin{bmatrix} t - t_0 \\ P - P_0 \end{bmatrix}$$

$$\begin{aligned} t' &= t_0' + \alpha(t - t_0) + b(P - P_0) \\ P' &= P_0' + c(t - t_0) + Q(P - P_0) \end{aligned}$$

Kdaj g ohranja časovnost?

$$(t_1, P_1) \mapsto (t_1', P_1')$$

$$(t_2, P_2) \mapsto (t_2', P_2')$$

$$t_2 - t_1$$

$$\begin{aligned} t_2' - t_1' &= (t_0' + \alpha(t_2 - t_0) + b(P_2 - P_0)) - (t_0' + \alpha(t_1 - t_0) + b(P_1 - P_0)) = \\ &= \alpha(t_2 - t_1) + b(P_2 - P_1) = \\ &= t_2 - t_1 \end{aligned}$$

↑
ohranja časovnost
(ker je a linear)
kdaj?

$$\begin{aligned} b &= 0 \\ \alpha &= 1 \end{aligned}$$

• Kdaj ohranja razdaljo med istočas. dog.?

$$(t_1, P_1) \mapsto (t_1', P_1')$$

$$(t_1, P_2) \mapsto (t_1', P_2')$$

$$\|P_2' - P_1'\| = \|P_2 - P_1\|$$

$$\begin{aligned} P_2' - P_1' &= (P_0' + c(t - t_0) + Q(P_2 - P_0)) - (P_0' + c(t - t_0) + Q(P_1 - P_0)) = \\ &= Q(P_2 - P_1) \end{aligned}$$

↳ mešičava, ki ohranja velikost

$$\begin{aligned} Q &= \text{ortogonalna} \\ Q &\in O(3) \end{aligned}$$

$$SO(3) = \text{rotacije}$$

⇒ rek

Galilejeva transformacija med dvema
naravnima Gal. strukturama na $\mathbb{R} \times \mathbb{E}$ je
oblike

$$\begin{aligned} t &\mapsto t' = t + t_0' \\ P &\mapsto P' = P_0' + \vec{c}t + Q(P - P_0) \end{aligned}$$

Parametri so: Q, c, t_0', P_0'
10 param. 3 3 1 3

$$A \in \mathbb{R} \times \mathbb{E}^3$$

$$A = (t, P)$$

$$g: \mathbb{R} \times \mathbb{E}^3 \rightarrow \mathbb{R} \times \mathbb{E}^3$$

Def: Galilejev koordinatni sistem je bijekcija $\varphi = (\tau, \pi): A \rightarrow \mathbb{R} \times \mathbb{E}$ tako, da je τ afina preslikava.

Trditev Galilejev koordinatni sistem inducira na A Galilejevo strukturo.

Dokaz $\varphi = (A, t, p)$, $t \in \mathcal{L}(V, \mathbb{R})$
 $A_1, A_2 \in A$
 $t(A_1, A_2) = ?$

$$\varphi(A_2) = (t_2, p_2)$$

$$\varphi(A_1) = (t_1, p_1)$$

$$t(A_1, A_2) := t_2 - t_1$$

$$\text{proj}_1(\varphi(A_2)) = \varphi(A_1)$$

$$\boxed{\text{proj}_1: \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{R}}$$

$$(t, P) \mapsto t$$

$$\text{proj}_1(\varphi(A_2)) - \text{proj}_1(\varphi(A_1))$$

$$= \tau(A_2) - \tau(A_1) = S(A_2 - A_1)$$

ker τ afina

$$S(A_1, A_2) = \|P_2 - P_1\|$$

koordinatni sistem je bijekcija
 $\varphi: A \rightarrow \mathbb{R} \times \mathbb{E}$

tokj mi en sam: $\tilde{\varphi}: A \rightarrow \mathbb{R} \times \mathbb{E}$

$\varphi \circ \varphi^{-1}$ ker je grupa $\tilde{\varphi} \circ \varphi^{-1}$, $(\tilde{\varphi} \circ \varphi^{-1})^{-1}$
 $\downarrow$ Gal. transform.
 $\varphi \circ \varphi^{-1}$ je Gal. transform. $\tilde{\varphi} \circ \varphi^{-1}$ je Gal. transform.

Def: Koordinatna sistema $\varphi, \tilde{\varphi}$ sta ekvivalentni, če je $\tilde{\varphi} \circ \varphi^{-1}$ Galilejeva transformacija.

$$A \xrightarrow{\varphi} \mathbb{R} \times \mathbb{E} \xleftarrow{\tilde{\varphi}} A$$

$$\tilde{\varphi} \circ \varphi^{-1}: \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{R} \times \mathbb{E}$$

Množica koord. sist. KS razpade na množico ekvivalentnih razredov.

$$\varphi, \tilde{\varphi} \text{ ekvivalentna}$$

$$(A, t, p) \downarrow (A, \tilde{t}, \tilde{p})$$

$$\Downarrow \text{Gal. strukt. tudi ekvivalentni}$$

Trditev Koordinatna sistema iz istega razreda Gal. KS inducirata ekvivalentno Gal. strukturo.

Dokaz Dopolaka $A_1, A_2 \in A$ $\underline{t(A_2 - A_1)} = \underline{\tilde{t}(A_2 - A_1)}$ (1)
 A_1, A_2 istočasna (v obeh) $\Rightarrow \underline{S(A_1, A_2)} = \underline{\tilde{S}(A_1, A_2)}$ (2)

$$(1) \quad t(A_2, A_1) = \text{proj}_1(\varphi(A_2)) - \text{proj}_1(\varphi(A_1)) = \text{proj}_1(\varphi(A_2) - \varphi(A_1)) = *$$

$$* \quad \varphi(A_2) = (\varphi \circ \tilde{\varphi}^{-1}) \circ \tilde{\varphi}(A_2)$$

$$* = \text{proj}_1(\varphi \circ \tilde{\varphi}^{-1})(\tilde{\varphi}(A_2) - \tilde{\varphi}(A_1)) = \tilde{t}_2 - \tilde{t}_1 = \tilde{t}(A_2 - A_1) \quad \checkmark$$

$\varphi \circ \tilde{\varphi}^{-1}$ Gal. transformacija $\rightarrow$ omenjena časovnost

$$\varphi(A_1) = (t_1, P_1), \quad \tilde{\varphi}(A_1) = (\tilde{t}_1, \tilde{P}_1) \Rightarrow t_2 - t_1 = \tilde{t}_2 - \tilde{t}_1$$

$t: P \mapsto P(t)$ temu medpisu pravimo trajektorija
Tir je zaloge vrednosti trajektorije.

VEKTOR HITROSTI = odvod položaja po času

$$\vec{v}(t) = \frac{dP(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = \dot{P}$$

VEKTOR POSPEŠKA = odvod vektora hitrosti

$$\vec{a} = \frac{d\vec{v}(t)}{dt} = \dot{\vec{v}} = \ddot{P}$$

sistem materialnih točk

$$P_1, P_2, \dots, P_n$$

$$\underline{P} = (P_1, \dots, P_n)$$

$$\dot{\underline{P}} = (\dot{P}_1, \dots, \dot{P}_n) \in \mathbb{R}^{3n}$$

$$\ddot{\underline{P}} = (\ddot{P}_1, \dots, \ddot{P}_n)$$

Na $\mathbb{R} \times \mathbb{E}$ imamo dve ekvivalentni strukturi:

$$(t, P) \text{ (opazov.)} \quad (t', P') \text{ (opazovalec')}$$

$$t' = t_0' + t$$

$$P' = P_0' + \vec{c}t + Q(P - P_0)$$

$$t \mapsto P(t); \quad P' = P'(t) = P_0' + \vec{c}t + Q(P(t) - P_0)$$

$$t' \mapsto P'(t')$$

$$t' \mapsto P'(t') = P_0' + \vec{c}(t - t_0') + Q(P(t - t_0') - P_0)$$

opazovalec

$$t \mapsto P(t)$$

$$\vec{v} = \frac{dP}{dt}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{P}$$

opazovalec'

$$t' \mapsto P'(t')$$

$$\vec{v}' = \frac{dP'}{dt'} = \vec{c} + Q\left(\frac{dP}{dt}(t)\right) = \vec{c} + Q\vec{v}$$

$$\vec{a}' = \dot{\vec{v}}' = Q\vec{a}$$

$$\boxed{\ddot{P}' = Q\ddot{P}}$$

Princip determiniranosti

Trajektorija $t \mapsto P(t)$ sistema materialnih točk je v danim KS natančno določena z začetnim položajem in hitrostjo.

To pomeni, da obstaja ^{→ določ. točk pospešila} funkcija interakcije f , tako da je $\ddot{P} = f(t, P, \dot{P})$.

Princip relativnosti

Obstaja tak razred ekvivalentnih Gal. KS, da je v tem razredu funkcija interakcije invariantna za Gal. transf. KSistemom iz tega razreda namreč inercialni koord. sistemi. $\Rightarrow \mathcal{G} = \mathcal{G}'$

KS	KS'
(t, P)	(t', P')
$\ddot{P} = f(t, P, \dot{P})$	$\ddot{P}' = f'(t', P', \dot{P}')$

$$\ddot{P}' = Q \ddot{P} = Q(\ddot{P}_1, \dots, \ddot{P}_n) = (Q(\ddot{P}_1), \dots, Q(\ddot{P}_n))$$

$$\dot{P}' = \dot{c} + Q \dot{P} = \dot{c} + Q(\dot{P}_1, \dots, \dot{P}_n) = (\dot{c} + Q\dot{P}_1, \dots, \dot{c} + Q\dot{P}_n)$$

$$f(t', P', \dot{P}') = Q \ddot{P} = Q f(t, P, \dot{P})$$

$$f(t_0' + t, P_0' + \dot{c}t + Q(P - P_0), \dot{c} + Q\dot{P}) = Q f(t, P, \dot{P})$$

Velja ^{ima 10 parametrov}

za vsako Gal. transf.

1. $Q = I, \dot{c} = \vec{0}, P_0' = P_0 : \Rightarrow f(t_0' + t, P, \dot{P}) = f(t, P, \dot{P}) \quad \forall t_0'$
 $\Rightarrow f = f(P, \dot{P})$ ^{odv. samo od položaja in hitrosti}
Homogenost časa

2. $Q = I, \dot{c} = \vec{0}, P_0' = P_0 + \vec{d} : P_0' + \dot{c}t + Q(P - P_0) = P_0 + \vec{d} + P - P_0 = P + \vec{d}$
 $\Rightarrow f(P + \vec{d}, \dot{P}) = f(P, \dot{P})$

$$f(P_1 + \vec{d}, P_2 + \vec{d}, \dot{P}_1, \dot{P}_2) = f(P_1, P_2, \dot{P}_1, \dot{P}_2)$$

$$(P_2 + \vec{d}) - (P_1 + \vec{d}) = P_2 - P_1 \quad \text{ni neodv. od položaja}$$

$$\Rightarrow f = f(P_i - P_j, \dot{P}) \quad f \text{ ^{ne}odvisna od relativnih položajev}$$

= HOMOGENOST PROSTORA

(3) $Q = 1, \vec{c};$

$$f(\vec{P}_i - \vec{P}_j, \vec{c} + \dot{\vec{P}}) = f(\vec{P}_i - \vec{P}_j, \dot{\vec{P}})$$

f je odvisna samo od relativne hitrosti

položaj + hitrost = faze prostor

(2) + (3) = homogenost faznega prostora

(4) $Q, \vec{c}:$

$$f(Q(\vec{P}_i - \vec{P}_j), Q(\dot{\vec{P}}_i - \dot{\vec{P}}_j)) = Q f(\vec{P}_i - \vec{P}_j, \dot{\vec{P}}_i - \dot{\vec{P}}_j)$$

f je IZOTOPICNA funkcija (vektorska)

= vse smeri prostora so enake

Posebni primer: $N=3$

$N=2$

$$\ddot{\vec{P}}_1 = \vec{f}_1(\vec{P}_2 - \vec{P}_1, \vec{P}_2 - \vec{P}_1)$$

$$\ddot{\vec{P}}_2 = \vec{f}_2(\vec{P}_2 - \vec{P}_1, \vec{P}_2 - \vec{P}_1)$$

$N=1$

$$\ddot{\vec{P}}_1 = \vec{f}_1 \text{ (ni argumentov)}$$

$$Q\vec{f}_1 = \vec{f}_1 \quad \forall Q \in O(3)$$

$$\Rightarrow \vec{f}_1 = \vec{0}$$

system ene mater. tocke

$$\Rightarrow \ddot{\vec{P}}_1 = \vec{0}$$

$\Rightarrow$ točka P_1 se giblje nemočrtno s konstantno hitrostjo

mesta materialna točka

= prvi Newtonov zakon

Posledica

$N=1$ nam da I Newtonov zakon.

$\forall$ KS se mesta točka giblje nemočrtno s konstantno hitrostjo.

$$\vec{F} = m\vec{a} \text{ ; pojem mase = ?}$$

PRINCIP SORAZMERNOSTI

Za poljubni mat. točki P_i in P_j iz mn. mat. točk P , ki tvorita zaprti sistem, obstaja pozitivna konstanta k_{ij} tako, da velja

$$\ddot{\vec{P}}_i = -k_{ij} \vec{P}_j \text{ (ne glede na vrsto interakcije med } P_i \text{ in } P_j)$$

za konstante k_{ij} velja $k_{ij}k_{jk}k_{ki} = 1$.

$$\left. \begin{aligned} \ddot{P}_i &= -k_{ji} \ddot{P}_j \\ \ddot{P}_j &= -k_{ij} \ddot{P}_i \end{aligned} \right\} \Rightarrow k_{ij} = \frac{1}{k_{ji}}$$

Lema Naj za konstante k_{ij} vega:

1) $k_{ij} > 0$

2) $k_{ij} = \frac{1}{k_{ji}}$

3) $k_{ij} \cdot k_{jk} \cdot k_{ki} = 1$

Potem obstajajo pozitivne konstante m_i tako, da je ~~to~~ ^(masa)

$$k_{ji} = \frac{m_j}{m_i}$$

Dokaz $\ell_{ij} := \log k_{ij}$ definiramo

$$\ell_{ij} = -\ell_{ji}$$

$$\ell_{ij} + \ell_{jk} + \ell_{ki} = 0$$

$$- \ell_{ioj} + \ell_{jki} + \ell_{kio} = 0$$

$$\ell_{ij} - \ell_{ioj} + \ell_{ki} - \ell_{kio} = 0$$

$$\ell_{ij} - \ell_{ioj} = -\ell_{ki} + \ell_{kio} = \ell_{ik} - \ell_{iok} \quad \forall j, k$$

$$\Rightarrow \ell_{ij} - \ell_{ioj} = m_{iio} \quad \text{neodvisen od } j$$

$$\ell_{ij} = m_{iio} + \ell_{ioj}$$

$$i=j \Rightarrow \ell_{ii} = m_{iio} + \ell_{ioi}$$

$$\Rightarrow \ell_{ioi} = -m_{iio}$$

$$\ell_{iio} = m_{iio}$$

$$\left. \begin{aligned} \ell_{ij} &= m_{iio} - m_{jio} \\ \text{definiramo } m_{iio} &:= \log m_i \end{aligned} \right\} \Rightarrow \ell_{ij} = \log m_i - \log m_j = \log \frac{m_i}{m_j}$$

$$\log k_{ij} = \log \frac{m_i}{m_j} \Rightarrow \underline{\underline{k_{ij} = \frac{m_i}{m_j}}}$$

Posledica $\ddot{P}_i = -k_{ji} \ddot{P}_j = -\frac{m_j}{m_i} \ddot{P}_j$

$$\boxed{m_i \ddot{P}_i = -m_j \ddot{P}_j}$$

Števila m_i pravimo inercialna masa.

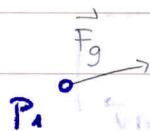
Produkt masa $\times$ pospešek =: sila.

* $\Leftarrow$

$$\left\{ \begin{aligned} \ddot{P}_i &= \vec{f}_i(P_i - P_j, \dot{P}_i - \dot{P}_j) \\ m_i \ddot{P}_i &= m_i \vec{f}_i = \vec{F}_i = \text{sila } P_j \text{ na } P_i =: \vec{F}_{ji} \\ \cancel{m_j \ddot{P}_j} &= \vec{F}_j \\ m_j \ddot{P}_j &= \cancel{m_j \ddot{P}_j} = \text{sila } P_i \text{ na } P_j =: \vec{F}_{ij} \\ \hline m_i \ddot{P}_i &= -m_j \ddot{P}_j \end{aligned} \right.$$

$$* \Rightarrow \vec{F}_{ij} = -\vec{F}_{ji} \quad \text{III. Newtonov zakon}$$

zveza med gravitacijsko in inercialno maso: enaki sta



P_2

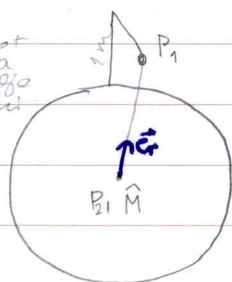
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masa

$$\vec{F}_g = \frac{\gamma \hat{m}_1 \hat{m}_2}{|P_1 - P_2|^2} \cdot \frac{P_1 - P_2}{|P_1 - P_2|} = \frac{\gamma \hat{m}_1 \hat{m}_2}{r^2} \vec{e}_r$$

$$\vec{r} = P_1 - P_2$$

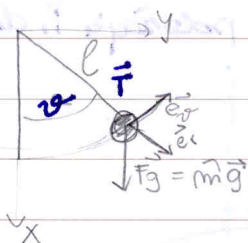
$$\vec{e}_r = \frac{\vec{r}}{|\vec{r}|}$$

se obnaša kot
bi bila vsa
masa zvezo
v eni točki



$$\vec{F}_g = -\frac{\gamma \hat{m}_1 \hat{M}}{r^2} \vec{e}_r \doteq \hat{m}_1 \vec{g} \quad \vec{g} \dots \text{težnostni pospešek}$$

$$\Rightarrow \vec{F}_g = \hat{m}_1 \vec{g}$$



sinusovica

$$m\vec{a} = \vec{F} = \hat{m} \vec{g} + \vec{T}$$

$$\vec{r} = r \vec{e}_r \quad ; \quad \vec{e}_r = \cos \vartheta \vec{i} + \sin \vartheta \vec{j}$$

$$\dot{\vec{r}} = \frac{d}{dt}(l \vec{e}_r) = l \frac{d\vec{e}_r}{d\vartheta} \cdot \frac{d\vartheta}{dt}$$

$$\frac{d\vec{e}_r}{d\vartheta} = -\sin \vartheta \vec{i} + \cos \vartheta \vec{j} = \vec{e}_\vartheta$$

$$\dot{\vec{r}} = l \dot{\vartheta} \vec{e}_\vartheta$$

$$\vec{a} = \ddot{\vec{r}} = l \ddot{\vartheta} \vec{e}_\vartheta + l \dot{\vartheta}^2 \frac{d\vec{e}_\vartheta}{d\vartheta}$$

$$\frac{d\vec{e}_\vartheta}{d\vartheta} = -\cos \vartheta \vec{i} - \sin \vartheta \vec{j} = -\vec{e}_r$$

$$\vec{a} = l \ddot{\vartheta} \vec{e}_\vartheta - l \dot{\vartheta}^2 \vec{e}_r$$

$$\vec{T} = -T \vec{e}_r$$

$$\vec{g} = g \vec{i} = g(\cos \vartheta \vec{e}_r - \sin \vartheta \vec{e}_\vartheta)$$

$$m(l \ddot{\vartheta} \vec{e}_\vartheta - l \dot{\vartheta}^2 \vec{e}_r) = \hat{m} g(\cos \vartheta \vec{e}_r - \sin \vartheta \vec{e}_\vartheta) - T \vec{e}_r$$

$$m l \ddot{\vartheta} = -\hat{m} g \sin \vartheta$$

$$-m l \dot{\vartheta}^2 = \hat{m} g \cos \vartheta - T$$

$$\ddot{\vartheta} + \frac{\hat{m} g}{m l} \sin \vartheta = 0$$

$$\sin \vartheta \doteq \vartheta$$

$$\omega^2 = \frac{\hat{m} g}{m l}$$

$$\ddot{\vartheta} + \omega^2 \vartheta = 0$$

enacha harmoničnega nihanja

$$\text{rešitev: } \vartheta = A \cos \omega t + B \sin \omega t = C \cos(\omega t - \delta)$$

ω = frekvenca

C = amplituda

δ = faza zamika

$$(A = C \cos \delta, B = C \sin \delta)$$

$$T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{\hat{m} g}{m l}}} \quad \text{perioda nihanja}$$

medu. od odklona

za vse možne eksperimente je Newton ugotovil, da je $\frac{m}{m_0}$ konstanten, ne glede na izbiro materiala (mihala)

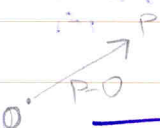
$$m = m_0$$

Enakost in. mase in grav. mase je eksperimentalno dokazano.

Vrtilna količina

$$\vec{L}(O) = (\vec{P} - \vec{O}) \times m \vec{\dot{P}}$$

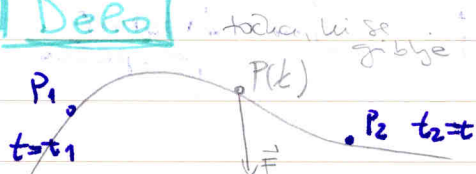
$\vec{P} - \vec{O}$ POL vrt. toč.
 $m \vec{\dot{P}}$ gib. količina



Gibalna količina

$$P, m \quad m \vec{\dot{P}} = m \vec{v}$$

Deło



Def. Deło sile $\vec{F}$ na mat. točko pri gibanju od položaja P_1 do P_2 je

$$A = \int_{P_1}^{P_2} \vec{F} d\vec{P} = \int_{t_1}^{t_2} \vec{F} \frac{d\vec{P}}{dt} dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{\dot{P}} dt$$

$$A = \int_{t_1}^{t_2} \vec{F} \cdot \vec{\dot{P}} dt \stackrel{\vec{F} = m \vec{\ddot{P}}}{=} m \int_{t_1}^{t_2} \vec{\ddot{P}} \cdot \vec{\dot{P}} dt = \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt} |\vec{\dot{P}}|^2 dt \Rightarrow A = T_2 - T_1$$

$\vec{\dot{P}} \cdot \vec{\dot{P}} = \frac{d}{dt} \left(\frac{1}{2} \vec{\dot{P}} \cdot \vec{\dot{P}} \right) = \frac{d}{dt} \left(\frac{1}{2} v^2 \right)$

$$\frac{1}{2} m |\vec{\dot{P}}|^2 = \frac{1}{2} m v^2 = T \quad \text{kinetična energija}$$

Izrek o dełu

Deło je enako spremembi kinetične energije.

$$A = \int_{P_1}^{P_2} \vec{F} d\vec{P} = T_2 - T_1$$

Pot. energija

Def Sile $\vec{F}$ je potencialna, če obstaja potencial $U = U(P, \vec{\dot{P}}, t)$, tako da je $\vec{F} = -\text{grad}_P U = -\left(\frac{\partial U}{\partial \vec{P}} \right)_{\text{translacijsko}}$

vektor = stolpec

skal. funkcija
 $P \mapsto U \in \mathbb{R}$

$\frac{\partial U}{\partial \vec{P}} \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$
funkcional
kovektor = vrstica

⇒

Def Potencialna sila je komzervativna, če je odvisna samo od položaja. ($U = U(P)$).

$$\vec{F} \cdot d\vec{P} = \left(\frac{\partial U}{\partial \vec{P}} \right)^T \cdot \vec{P} dt = - \frac{dU}{d\vec{P}} \vec{P} dt$$

skal. mod. mod. vekt. stopnja

$$U = U(P)$$

$$\frac{d}{dt} U(P(t)) = \frac{dU}{d\vec{P}} \cdot \vec{P} \Rightarrow \vec{F} \cdot d\vec{P} = - \left(\frac{dU}{dt} \right) dt = -dU$$

$$A = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{P} = \int_{P_1}^{P_2} -dU = -(U_2 - U_1) \quad ; \quad U_2 = U(P_2), U_1 = U(P_1)$$

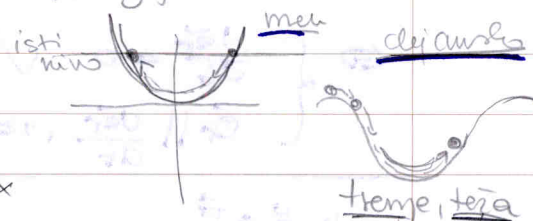
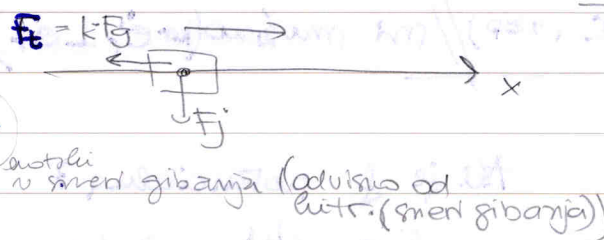
$$\Rightarrow \left. \begin{array}{l} \vec{F} \text{ komzervativna} \rightarrow A = U_1 - U_2 \\ A = T_2 - T_1 \end{array} \right\} \Rightarrow \begin{array}{l} T_2 - T_1 = U_1 - U_2 \\ U_2 + T_2 = U_1 + T_1 \end{array}$$

zrek (o energiji) Če je sila komzervativna, je vsota kinetične in potencialne energije konstanta gibanja.

$$(T + U = E_0) \quad \begin{array}{l} \text{energ. s. obrnjenja} \\ \text{mehaniška energija} \end{array}$$

trnje

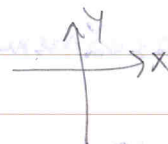
$$\vec{F}_t = -kF_g \cdot \frac{\vec{x}}{|\vec{x}|}$$



mi komzervativna (odvisno od smeri gibanja ne od položaja)

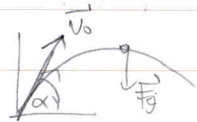
je potencialna

$$\begin{array}{l} F_g = -mg \cdot \vec{g} \\ U = mgy + U_0 \\ \text{odl. od položaja} \end{array}$$



Premočrtno gibanje (gibanje z eno prostostno stopnjo)

poševni met



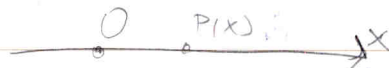
ekviv. po Gal. transl. z U_E op. 2. iv op. 1
vidita drug. tir.

$$\vec{v}_0 = v_0 (\cos \alpha \vec{i} + \sin \alpha \vec{j})$$

- ① opazovalec 1 v izhodišču: ~~tir~~ vidi: tir = parabola
- ② opazovalec 2 se giblje v smeri x-os z hitr. v_0 : tir = premica (prosti pad)

Trditev Gibanje je premočrtno (v danem KS) $\Leftrightarrow$ obstaja točka O tako, da je $\vec{r}(0) = \vec{0}$.

Dokaz $\rightarrow$: $\vec{r}(0) = (P-O) \times m \vec{p}$ vrt. količina



premočrtno gibanje $\Leftrightarrow \vec{p}$ ima konstantno smer

O katnakoli točka na premici

$$\Leftrightarrow: \vec{r}(0) = (P-O) \times m \vec{p}$$

$$\vec{r} = r \cdot \vec{e}_r, |\vec{e}_r| = 1$$

$$\vec{p} = \dot{\vec{r}} = \frac{d}{dt}(r \vec{e}_r) = \dot{r} \vec{e}_r + r \frac{d\vec{e}_r}{dt}$$

$$\vec{r}(0) = (r \vec{e}_r) \times m (\dot{r} \vec{e}_r + r \frac{d\vec{e}_r}{dt}) = m r^2 \vec{e}_r \times \frac{d\vec{e}_r}{dt} = \vec{0} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{d\vec{e}_r}{dt} = 0 \quad \checkmark \\ \vec{e}_r \parallel \frac{d\vec{e}_r}{dt} \text{ (vzp)} // \text{ ni možno (ker } \vec{e}_r \perp \frac{d\vec{e}_r}{dt} \end{cases}$$

$$m \ddot{\vec{a}} = \vec{F}$$

$$\vec{a} = \ddot{\vec{x}} = \ddot{x} \vec{i}$$

$$\vec{F} = f \vec{i}$$

$$m \ddot{x} = f$$

Ki je f potencialna?

$$f = -\frac{dU}{dx} \Leftrightarrow U = -\int_{x_0}^x f(\xi) d\xi + U_0, \text{ če } f \text{ zvezna}$$

$\Rightarrow$ V enorazsežnem primeru je vsaka zvezna sila potencialna.

Trditev Naj bo $\vec{F} = f(x) \vec{i}$. Če je f zvezna funkcija, je $\vec{F}$ konzervativna sila. Potem je velja izrek o energiji.

$$T + U = E_0$$

$$\boxed{\frac{1}{2} m \dot{x}^2 + U(x) = E_0}$$



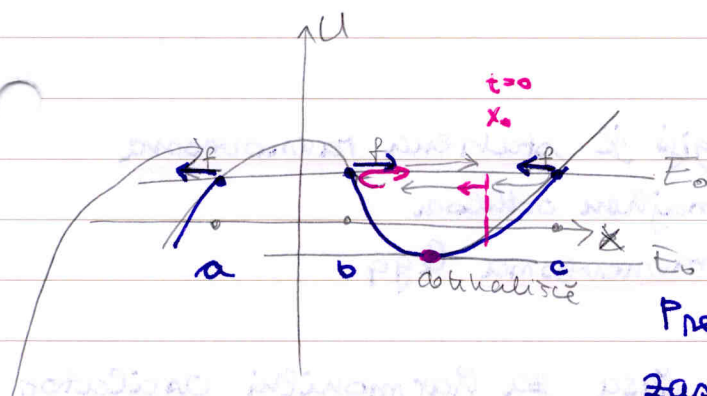
$$\dot{x}^2 = \frac{2}{m} (E_0 - U(x)) \Rightarrow \dot{x} = \pm \sqrt{\frac{2}{m} (E_0 - U(x))} = \frac{dx}{dt} \quad \text{diferencialna}$$

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m} (E_0 - U(x))}} = \int_{t_1}^{t_2} (\operatorname{sgn} \dot{x}) dt$$

$$(\operatorname{sgn} \dot{x}) \cdot \int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m} (E_0 - U(x))}} = t_2 - t_1 \quad t = t(x) \\ \Rightarrow x = x(t)$$

Premočrtno gibanje pod vplivom konzervativne sile je integrabilno.

Kvalitativna obravnava rešitve



Gibanje je možno samo tam, kjer je energ. ravnjmica nad potencialom.

Presečišča energ. niv. in potenciala so zastojne točke (tj. $\dot{x} = 0$)
"točke miniranja"

To so točke trenutnega mirovanja.
(za trenutek se ustavi, nato nadaljuje pot v smeri f)

To so točke obrata.

$$f = -\frac{dU}{dx}$$

Gibanje na intervalih je periodično.

Omejeno p.i. gibanje je periodično.

dotikalisci:

V dotikalisku je $\dot{x} = 0$ in $f = 0 \Rightarrow$ je točka miniranja.

Gibanje na poltraku je neomejeno.

perioda:

$$-\int_{x_0}^b \frac{dx}{\sqrt{\frac{2}{m} (E_0 - U(x))}} + \int_b^c \frac{dx}{\sqrt{\frac{2}{m} (E_0 - U(x))}} - \int_c^{x_0} \frac{dx}{\sqrt{\frac{2}{m} (E_0 - U(x))}} = T(\text{čas})$$

$$T = 2 \int_b^c \frac{dx}{\sqrt{\frac{2}{m} (E_0 - U(x))}}$$

V krajišču je imenovalec enak 0.

Ali sploh obstaja?

↑ pospe. integral Ali je konvergenten (omejen)?

$$\text{V okolici } x=b: U(x) = U(b) + \frac{dU}{dx} \Big|_{x=b} (x-b) + O((x-b)^2)$$

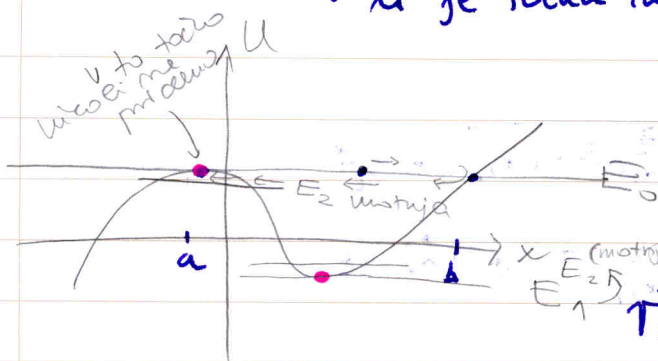
$$E_0 - U(x) = -\frac{dU}{dx} \Big|_{x=b} (x-b) + O((x-b)^2)$$

Enako v c krajišču!

Alketa

$$\frac{dx}{\sqrt{E_0 - U(x)}} \doteq \frac{dx}{\sqrt{x-b}} \quad \text{obstaja?} \\ \int_0^1 \frac{dx}{\sqrt{x-b}} < 1 \quad \text{obstaja}$$

v je točka mirovanja



V okolici $x=a$:

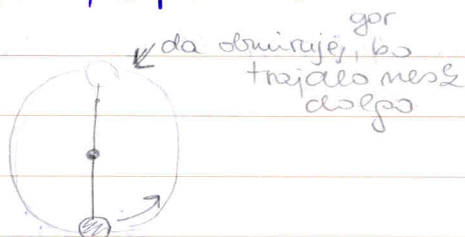
$$U(x) = U(a) + \frac{dU}{dx}\bigg|_{x=a}(x-a) + \frac{1}{2} \frac{d^2U}{dx^2}\bigg|_{x=a}(x-a)^2 + \dots$$

$$\frac{dx}{\sqrt{E_0 - U(x)}} = \frac{dx}{\sqrt{\frac{1}{2} \frac{d^2U}{dx^2}\bigg|_{x=a}(x-a)^2}}$$

ne konvergira

V točko mirovanja ne prideemo v končnem času. Gibanje ni periodično.

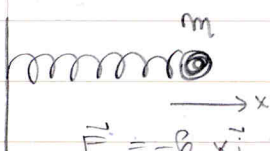
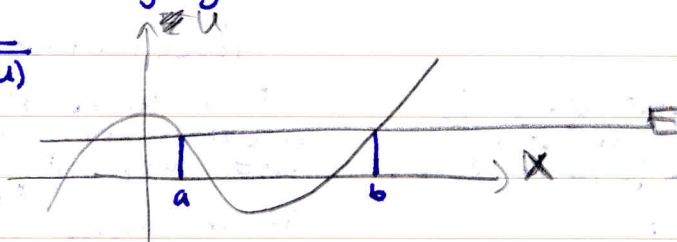
KAT. NIHALO:



Lokalni minimum pot. energije je stabilna ravnovesna lega. Majhna ^{motnja} povzroči majhen odklon.
Lokalni max. je nestabilna ravnovesna lega.

Primer Izračun nihajnega časa za harmonični oscilator.

$$T = 2 \int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E - U)}}$$



$$\vec{F} = -k \vec{x}$$

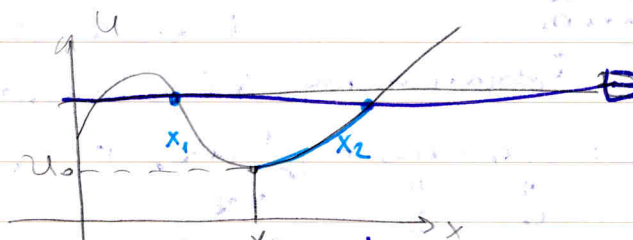
$$U = \frac{1}{2} k x^2$$

na voljo

$$T = 2\pi \sqrt{\frac{m}{k}}$$

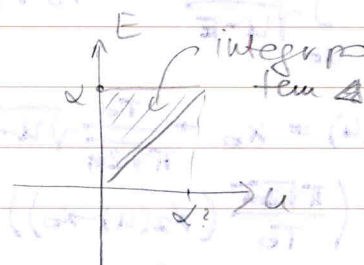
Izokronično nihanje $\Leftrightarrow$ nihajni čas je neodvisen od energije

iz dane zveze $T = T(E)$ iščemo potencial.

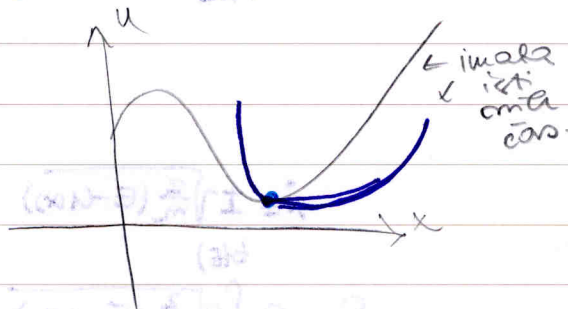


$$T = \sqrt{2m} \int_{a(E)}^{b(E)} \frac{dx}{\sqrt{E - U}} = \sqrt{2m} \left(\int_a^{x_0} \frac{dx}{\sqrt{E - U}} + \int_{x_0}^b \frac{dx}{\sqrt{E - U}} \right) =$$

$$\begin{aligned}
 dx &= \frac{dx_1}{du} du \\
 &= \sqrt{2m} \left(\int_{E_0}^u \frac{\frac{dx_1}{du} du}{\sqrt{E-u}} + \int_u^E \frac{\frac{dx_2}{du} du}{\sqrt{E-u}} \right) = \sqrt{2m} \int_{u_0}^E \frac{1}{\sqrt{E-u}} \left(\frac{dx_2}{du} - \frac{dx_1}{du} \right) du \quad \left| \frac{1}{\sqrt{\alpha-E}} dE \right| \\
 &= \int_{u_0}^{\alpha} \frac{T(E) dE}{\sqrt{\alpha-E}} = \sqrt{2m} \int_{u_0}^{\alpha} \frac{dE}{\sqrt{\alpha-E}} \int_{u_0}^E \frac{1}{\sqrt{E-u}} \left(\frac{dx_2}{du} - \frac{dx_1}{du} \right) du = \text{zanemaj: prvni} \\
 &= \sqrt{2m} \int_{u_0}^{\alpha} du \int_u^{\alpha} dE \frac{1}{\sqrt{E-u}(\alpha-E)(E-u)} \left(\frac{dx_2}{du} - \frac{dx_1}{du} \right) = \text{molit} \\
 &= \int_{u_0}^{\alpha} \frac{T(E) dE}{\sqrt{\alpha-E}} = \sqrt{2m} \int_{u_0}^{\alpha} \left(\frac{dx_2}{du} - \frac{dx_1}{du} \right) du \int_u^{\alpha} \frac{dE}{\sqrt{(\alpha-E)(E-u)}} = \text{plošina} \\
 &= \int_{-1}^1 \frac{dz}{\sqrt{1-z^2}} = \arcsin z \Big|_{-1}^1 = \frac{\pi}{2} \\
 &= \pi \sqrt{2m} \int_{u_0}^{\alpha} \left(\frac{dx_2}{du} - \frac{dx_1}{du} \right) du = \pi \sqrt{2m} (x_2(\alpha) - x_1(u_0)) \\
 &= \pi \sqrt{2m} \left(\int_{x_0}^{x_2(\alpha)} dx - \int_{x_0}^{x_1(\alpha)} dx \right) = \pi \sqrt{2m} (x_2(\alpha) - x_0) - \pi \sqrt{2m} (x_1(\alpha) - x_0) \\
 &= \pi \sqrt{2m} (x_2(\alpha) - x_1(\alpha))
 \end{aligned}$$



Lemma V okolí stabilne rovnovesne křižky veje.



Definice Zveza

$T=T(E)$ ne določa evklidovskú potenciálu. Določa samo širinu potenciálu (rozdiel medzi x_2 a x_1).

Posledice Symetricki potenciál:

$$x_2 - x_0 = x_0 - x_1$$

$$x_2(u) - x_1(u) = x_2(u) + x_1(u) + 2x_0$$

$$= 2(x_2(u) - x_1(u))$$

$$x_2(u) = x_0 + \frac{1}{2\pi\sqrt{2m}} \int_{u_0}^u \frac{T(E) dE}{\sqrt{u-E}}$$

Če je potencial simetričen v okoliščini stabilne lege, je nataknjen določen z zvezo $T = T(E)$.
v tej okoliščini

Primer izohonično gibanje ($T(E) = T_0$ konst.)

$$\int_{u_0}^u \frac{T_0 dE}{\sqrt{u-E}} = T_0 (-2) \sqrt{u-E} \Big|_{u_0}^u = -2T_0 (0 - \sqrt{u-u_0}) = \underline{\underline{2T_0 \sqrt{u-u_0}}}$$

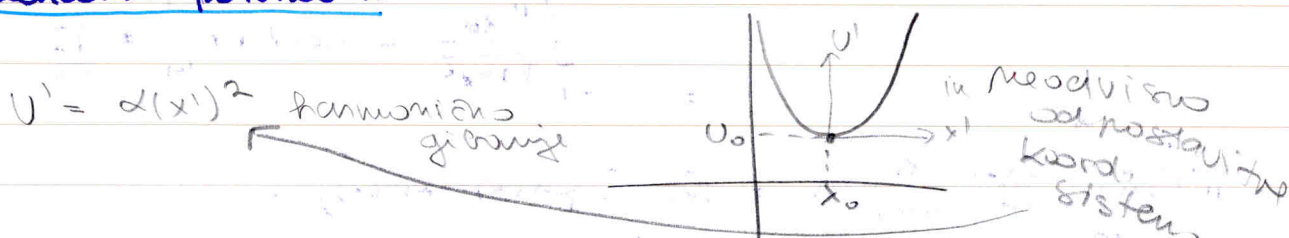
$$x_2(u) = x_0 + \frac{T_0}{\pi \sqrt{2m}} \sqrt{u-u_0}$$

$$u = \left(\frac{\pi \sqrt{2m}}{T_0} (x_2(u) - x_0) \right)^2 + u_0$$

$$u(x) = u_0 + \frac{\pi^2 2m}{T_0^2} (x - x_0)^2$$

kvadratični potencial

Izohonično gibanje s simetričnim potencialom ima kvadratični potencial.



Edino izohr.-gibanje s sim. potenc. je harmonično gibanje

reš. harm. enačbe:

$$\ddot{x} + \omega^2 x = 0$$

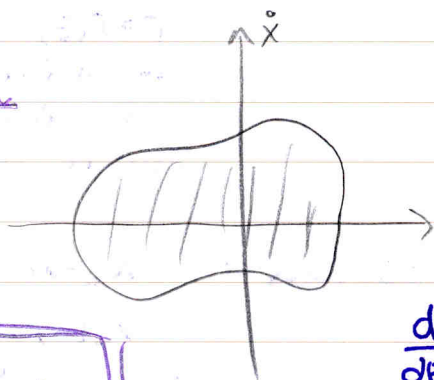
$$x = A \cos(\omega t - \delta)$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\alpha = \frac{\pi^2 2m}{T_0^2} = \frac{\pi^2 2m K}{4 \pi^2 m} = \frac{1}{2} K$$

~~razpis~~

Fazna razmerna



$$\dot{x} = \pm \sqrt{\frac{2}{m} (E - u(x))}$$

$b(E)$

$$S = 2 \int_a^{b(E)} \sqrt{\frac{2}{m} (E - u(x))} dx$$

$a(E)$

$$\frac{dS}{dE} = 2 \sqrt{\frac{2}{m}} \left(\frac{\partial b}{\partial E} \sqrt{E - u(b)} - \frac{\partial a}{\partial E} \sqrt{E - u(a)} + \int_a^b \frac{dx}{2 \sqrt{E - u}} \right)$$

$$u(a) = u(b) = E$$

$$\boxed{\frac{dS}{dE} = \sqrt{\frac{2}{m}} \int_a^b \frac{dx}{\sqrt{E - u}} = \frac{1}{m} T}$$

$T = \frac{1}{\omega}$

Traiter $\frac{ds}{dE} = \frac{1}{mT}$

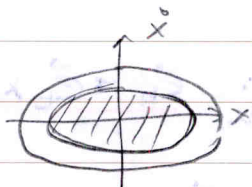
Primer harmonični oscilator = $U = \frac{1}{2}kx^2$

$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E$ energ. jednačina

$\frac{\dot{x}^2}{a^2} + \frac{x^2}{b^2} = 1$
elipsa

$\frac{\dot{x}^2}{\frac{2E}{m}} + \frac{x^2}{\frac{2E}{k}} = 1$

$a = \sqrt{\frac{2E}{m}}, b = \sqrt{\frac{2E}{k}}$



elipsa

$S = \pi ab = \pi \sqrt{\frac{2E}{m}} \sqrt{\frac{2E}{k}} = \pi \frac{2E}{\sqrt{mk}}$

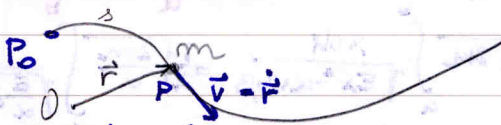
$\frac{ds}{dE} = \frac{1}{mT} \Rightarrow$

$T = 2\pi \frac{m}{\sqrt{mk}} = 2\pi \sqrt{\frac{m}{k}}$
vremena perioda

$\frac{ds}{dE} = 2\pi \frac{1}{\sqrt{mk}}$

GIBANJE PO KRIVULJI

Dana je krivulja



To je gibanje z eno prostorno stopnjom: s ... unavumi parameter "doležina"

$P = P(t)$

$\vec{r} = P - O$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt} \vec{r}(s(t)) = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} = \dot{s} \vec{e}_t$

$|\vec{v}| = |\dot{s}|$

$\frac{d\vec{r}}{ds} = \vec{e}_t$

enotski tangenčni vektor

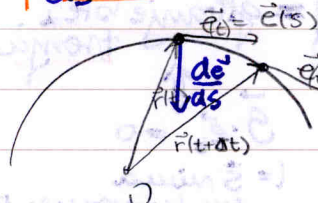
$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\dot{s}\vec{e}_t)}{dt} = \ddot{s}\vec{e}_t + \dot{s} \frac{d\vec{e}_t}{dt}$

$= \ddot{s}\vec{e}_t + \kappa \dot{s}^2 \vec{e}_n$

$\frac{d\vec{e}_t}{ds} = \kappa \vec{e}_n$

$|\vec{e}_n| = 1$
 $\kappa = \left| \frac{d\vec{e}_t}{ds} \right|$

κ = ukrivljenost
"1/R"



$\lim_{\Delta s \rightarrow 0} \frac{\vec{e}(s+\Delta s) - \vec{e}(s)}{\Delta s}$

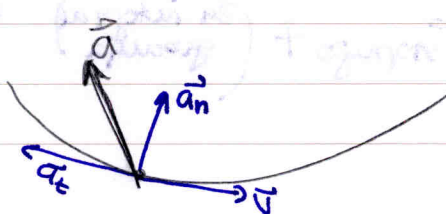
$\vec{a} = \vec{a}_t + \vec{a}_n$

$\vec{a}_t = \ddot{s}\vec{e}_t$

$\vec{a}_n = \kappa \dot{s}^2 \vec{e}_n$

tangentni pospešek

normalni pospešek



$\vec{a} \cdot \vec{v} < 0$ zaviranje
"s"

(=

$m\ddot{\mathbf{a}} = \vec{F} / \vec{e}_t \wedge \vec{e}_n$ $\vec{F}$ je vsota aktivnih zun. sil in sila vezi.

1. $m\ddot{s} = (\vec{F} \cdot \vec{e}_t) \Rightarrow s = s(t)$, če $\vec{F}$ tuarna
2. $\kappa \dot{s}^2 = (\vec{F} \cdot \vec{e}_n)$

$\mathbb{R}^3$ prostoru $\vec{e}_b = \vec{e}_t \times \vec{e}_n$ binormalna

$\vec{a} \cdot \vec{e}_b = 0 \Rightarrow 0 = \vec{F} \cdot \vec{e}_b$

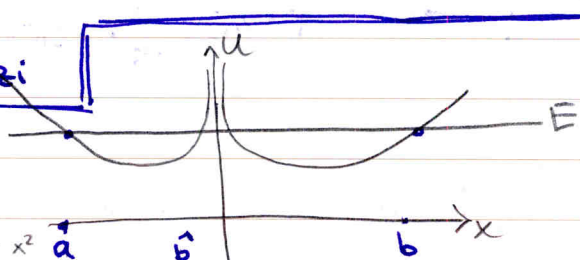
3 enačbe, 3 neznanke
 $\Rightarrow$ 2 komp. sile vezi

PRIMER
OD PRED

* $U(x) = \alpha x^2 + \frac{\beta}{x^2}$

perioda: $T = \int_a^b \sqrt{2m} \frac{dx}{\sqrt{E-U}} =$

$= \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E - \alpha x^2 - \frac{\beta}{x^2}}} = \sqrt{2m} \int_a^b \frac{x dx}{\sqrt{E x^2 - \alpha x^4 - \beta}} = \frac{1}{2} \sqrt{\frac{2m}{\alpha}} \int_a^b \frac{dz}{\sqrt{\frac{E}{\alpha} z^2 - z^2 - \frac{\beta}{\alpha}}}$



na krajščih a, b je enako nič
 $= \pi$

$T = \pi \sqrt{\frac{m}{2\alpha}}$ perioda T je neodvisna od E in β .

$\hookrightarrow$ izokronično gibanje z nesimetričnim potencialom

$m\ddot{\mathbf{a}} = \vec{F} + \vec{S}$
silna? upot? sila vezi? tuja (+trnje)?

Posebni primer gibanje po gladki krivulji (= gibanje brez trnja)

$A = \int_{P_1}^{P_2} (\vec{F} + \vec{S}) d\vec{P} = \int \vec{F} d\vec{P}$
 $\vec{S} \cdot \vec{e}_t = 0$

$\vec{S} \cdot \vec{e}_t = 0$
 (= $\vec{S}$ nima ene komponente)

$d\vec{P} = \frac{d\vec{P}}{ds} ds = \vec{e}_t ds$

Posledica Pri gibanju po gladki krivulji pod upirno komz. sil velja energijski izrek.

$\Downarrow$ gibanje z eno prostorsko stopnjo + en integral gibanja
 " ena konstanta gibanja"
 gibanje je integralno

$$\vec{a} = \ddot{s}\vec{e}_t + \dot{s}^2\vec{e}_n$$

$$m\vec{a} = \vec{F} + \vec{S} \quad / \cdot \vec{e}_t$$

$$m\ddot{s} = F \cdot \vec{e}_t$$

F konzervativna $\Rightarrow \vec{F} = -\text{grad } U, U = U(P).$

$$\frac{dU}{dt} = \frac{d}{dt} U(P(t)) =$$

$$\frac{dU}{ds} = \frac{d}{ds} U(P(t)) = \frac{dU}{dP} \frac{dP}{ds} = -\vec{F} \cdot \vec{e}_t$$

$$\hookrightarrow m\ddot{s} = - \frac{dU}{ds} \quad / \cdot \dot{s}$$

$$\frac{d}{ds} \left(\frac{1}{2} m \dot{s}^2 \right) + \frac{d}{dt} U = 0$$

$$\} \Rightarrow \boxed{\frac{1}{2} m \dot{s}^2 + U(s) = E_0} \quad \left(= \text{energijška} \right)$$

za
gibanje po kriv.
večja
energijška
učinka

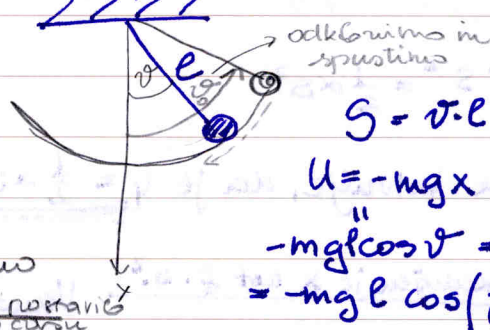
To je energijska enačba za memočno gibanje v konfiguracij. prostoru ~~na~~ ločne dolžine.

To pomeni, da smo gibanje po geodeti kroglji reducirali na memočno gibanje.

Primer matematično nihalo

gibanje "mratrenja" po krožnici

$$U = -mgl \cos \vartheta$$



$$S = r \cdot l$$

$$U = -mgx$$

$$-mgl \cos \vartheta =$$

$$= -mgl \cos \left(\frac{s}{l} \right)$$

$$\frac{1}{2} m \dot{s}^2 - mgl \cos \left(\frac{s}{l} \right) = E_0$$

$$T = ?$$

$$= 2\pi \sqrt{m} \int_0^{s_0} \frac{ds}{\sqrt{E_0 + mgl \cos \left(\frac{s}{l} \right)}} =$$

$$= 2\pi \sqrt{\frac{l}{g}} \int_0^{\vartheta_0} \frac{d\vartheta}{\sqrt{\cos \vartheta - \cos \vartheta_0}} \quad \text{polovni kot}$$

$$\left. \begin{aligned} 1 - \cos \vartheta &= 2 \sin^2 \frac{\vartheta}{2} \\ \cos \vartheta &= 1 - 2 \sin^2 \frac{\vartheta}{2} \end{aligned} \right\} \quad 1 - \cos \vartheta = 2 \sin^2 \frac{\vartheta}{2}$$

$$* 2\sqrt{\frac{l}{g}} \int_0^{\vartheta_0} \frac{d\vartheta}{\sqrt{1 - \sin^2 \frac{\vartheta}{2} - \sin^2 \frac{\vartheta_0}{2}}} = 2\sqrt{\frac{l}{g}} \frac{1}{\sin \frac{\vartheta_0}{2}} \int_0^1 \frac{du}{\sqrt{1 - u^2} \sqrt{1 - \sin^2 \frac{\vartheta_0}{2} u^2}}$$

$$\frac{1}{2} \cos \frac{\vartheta}{2} d\vartheta = \sin \frac{\vartheta_0}{2} du$$

$$T = 4\sqrt{\frac{l}{g}} \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-\sin^2 \frac{\vartheta_0}{2} u^2)}} = \text{eliptični integral}$$

$\text{Jedna za } |v_0| \ll 1: 1 - \sin^2 \frac{v_0}{2} u^2 < 1 \Rightarrow 4\sqrt{\frac{g}{g}} \int_0^1 \frac{du}{\sqrt{1-u^2}} < T$ arcsin / 1
 $0 \leq u \leq 1$ $2\pi\sqrt{\frac{g}{g}} < T$

$1 - \sin^2 \frac{v_0}{2} u^2, u=1 > 1 - \sin^2 \frac{v_0}{2} = \cos^2 \frac{v_0}{2}$
 $\Rightarrow 4\sqrt{\frac{g}{g}} \int_0^1 \frac{du}{\sqrt{(1-u^2)(\cos^2 \frac{v_0}{2})}} > T$
 $T < \frac{1}{\cos \frac{v_0}{2}} 2\pi\sqrt{\frac{g}{g}}$

$2\pi\sqrt{\frac{g}{g}} < T < \frac{1}{\cos \frac{v_0}{2}} 2\pi\sqrt{\frac{g}{g}}$

invarijant, 1

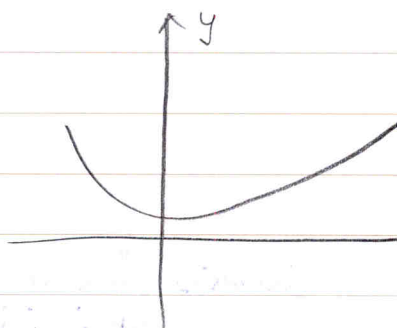
simetrično
 Ali obstaja izotonično nihalno (da je perioda neodvisna od energije)?

$\frac{1}{2}mv^2 + U(s) = E_0$

$U(s) = mgy = \frac{1}{2}\alpha s^2$

gibanje po gladini
 krogovi pod vplivom
 sile teže
 gibanje pod vpliv teže
 $\Rightarrow$ potencial $U = \dots$

$y = \frac{\alpha}{2mg} s^2 = \frac{1}{2}\alpha s^2$



Iščem tako krivulje, da je $y = \frac{1}{2}\alpha s^2$.

$y = y(s)$

$x = x(s)$? kakšen je x kot f. s^2

Vemo: $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$

$\frac{dx}{ds} = \sqrt{1 - (\alpha s)^2}$

Poravnano: $u = \sqrt{1 - \alpha^2 s^2}$ $du = \frac{-2\alpha s}{2\sqrt{1 - \alpha^2 s^2}} ds$
 $dx = ds$ $V = s$

$x = \int \sqrt{1 - (\alpha s)^2} ds =$

$= \cancel{x} \int \sqrt{1 - \alpha^2 s^2} ds + \int \frac{\alpha^2 s^2 + 1}{\sqrt{1 - \alpha^2 s^2}} ds \Rightarrow \frac{4}{3}$
 $\int \sqrt{1 - \alpha^2 s^2} ds - \int \frac{ds}{\sqrt{1 - \alpha^2 s^2}}$

$x = \frac{1}{2} \left(s \sqrt{1 - \alpha^2 s^2} + \frac{1}{2\alpha} \arcsin \alpha s \right)$; $y = \frac{1}{2} \alpha s^2 \Rightarrow s = \sqrt{\frac{2y}{\alpha}}$

$x = \frac{1}{2} \left(\sqrt{\frac{2y}{\alpha}} \sqrt{1 - 2\alpha y} + \frac{1}{2\alpha} \arcsin \sqrt{2\alpha y} \right) =$
 $\frac{1}{2\alpha} \arccos(1 - 4\alpha y)$

formula

$2 \arcsin z =$
 $= \arccos(1 - 2z^2)$

$1 - 4\alpha y = \cos 2\vartheta$

$y = \frac{1}{4\alpha} (1 - \cos 2\vartheta)$

$x = \frac{1}{2} \left(\sqrt{\frac{2y}{\alpha}} - 4y^2 + \frac{1}{2\alpha} 2\vartheta \right) = \left(\frac{1}{2\alpha^2} - \frac{1}{4\alpha^2} \right) + \cos 2\vartheta \left(-\frac{1}{2\alpha^2} + \frac{1}{2\alpha^2} \right) - \frac{1}{4\alpha^2} \cos^2 2\vartheta$
 $\frac{2y}{\alpha} - 4y^2 = \frac{1}{2\alpha^2} (1 - \cos 2\vartheta) - 4 \frac{1}{16\alpha^2} (1 - 2\cos 2\vartheta + \cos^2 2\vartheta) = \frac{1}{4\alpha^2} (1 - \cos^2 2\vartheta) = \frac{1}{4\alpha^2} \sin^2 2\vartheta$

$$\Rightarrow x = \frac{1}{2} \left(\frac{1}{2\alpha} \sin v + \frac{1}{2\alpha} v \right) = \frac{1}{4\alpha} (v + \sin v)$$

CIKLOIDA

$$x = a v + a \sin v = a(v + \sin v)$$

$$y = a - a \cos v = a(1 - \cos v)$$

$$\text{mas } a = \frac{1}{4\alpha}$$

Istakana krivulja je cikloida!

$$T = 4\pi \sqrt{\frac{a}{g}} \quad \text{perioda}$$

$$\alpha = \frac{g}{2}$$

$$U(s) = \frac{1}{2} \alpha s^2$$

$$T = 2\pi \sqrt{\frac{m}{k}} = \left(\text{harmon. oscilator} \right)$$

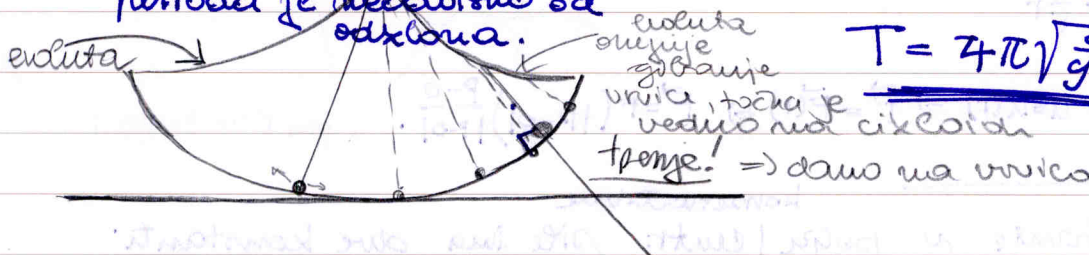
$$= 2\pi \sqrt{\frac{m}{\frac{1}{2} \alpha}} = 2\pi \sqrt{\frac{m}{\frac{1}{2} \frac{g}{2}}} = 2\pi \sqrt{\frac{4a}{g}}$$

$$T = 4\pi \sqrt{\frac{a}{g}}$$

Edino sim. izokron. mitalo je cikloidno mitalo.

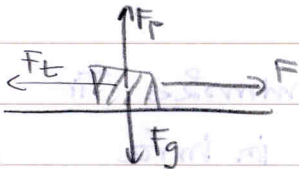
$$T = 4\pi \sqrt{\frac{a}{g}}$$

perioda je neovisno od odsklona.



Gibanje po nepravilnoj krivulji (= gibanje s trenjem)

$$m\vec{a} = \vec{F} + \vec{S} \quad ; \quad \vec{S} = S_t \vec{e}_t + S_n \vec{e}_n + S_b \vec{e}_b$$



$$|\vec{F}_t| = k |\vec{F}_p|$$

$$|S_b| = k \sqrt{S_n^2 + S_b^2} \quad ; \quad \dot{s} > 0$$

$$m\ddot{s} = F \vec{e}_t - k \sqrt{S_n^2 + S_b^2}$$

$$k\dot{s}^2 = F \vec{e}_n + S_n$$

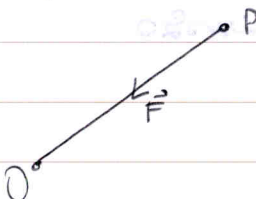
$$0 = F \vec{e}_b + S_b$$

nonlinearan sistem

nelinearna
resitev

Gibanje v polju centralne sile

Def Sila $\vec{F} = \vec{F}(P)$ je centralna, če obstaja točka O , tako da je $\vec{F} = \vec{F}(|P-O|) \frac{P-O}{|P-O|}$. Točki O pravimo center sile (ali pol sile). Npr: gravitacijska sila, sila elastične ...



Trditev: Konzervativna sila F je centralna $\Leftrightarrow \exists O$ tako, da je $\vec{L}(O, P) = \text{konst.}$

Dokaz: $\Rightarrow: \vec{L}(O, P) = (P-O) \times m\dot{P}$
 $\frac{d\vec{L}}{dt} = (P-O) \times m\ddot{P} \stackrel{\text{prvi}}{=} (P-O) \times \vec{F} \stackrel{P \times m\ddot{P} = 0?}{=} \vec{0} = (P-O) \times \vec{F} = \vec{0}$
 $\vec{F} \parallel (P-O).$

$\Leftarrow: \frac{d\vec{L}}{dt} = \vec{0}. (P-O) \times \vec{F} = \vec{0} \Rightarrow (P-O) \parallel \vec{F}$

$\vec{F}$ odvisen samo od razdalje $|P-O|$

$\vec{F} = \vec{F}(P) = -\text{grad } u$

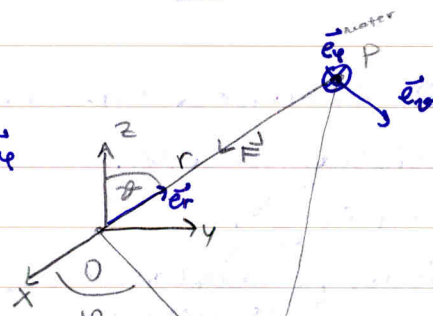
r, φ, ϑ krogele koordinate

$\text{grad } u = \frac{du}{dr} \vec{e}_r + \frac{1}{r} \frac{du}{d\vartheta} \vec{e}_\vartheta + \frac{1}{r \sin \vartheta} \frac{du}{d\varphi} \vec{e}_\varphi$

$\vec{e}_r = \frac{P-O}{|P-O|} = \frac{\vec{r}}{|\vec{r}|} = \frac{1}{r} \vec{F}$

$\vec{F} = -\text{grad } u$
 $\vec{F} \parallel \vec{e}_r$

$\Rightarrow u = u(r) \Rightarrow \vec{F} = \vec{F}(r) \Rightarrow \vec{F} = F(|P-O|) \frac{P-O}{|P-O|}$



Posledica ^{konzervativne} gibanje v polju centr. sile ima dve konstanti: gibanja, energijo in vrtilno količino.
 ① skalarna $\rightarrow$ 4 konst. ② vektorska $\rightarrow$ 3

lema: Gibanje v polju centralne sile je ravno. Tir leži v ravnini, ki vsebuje center sile in ima normalo v smeri vektorja vrtilne količine.

Dokaz: $(P(t)-O) \cdot \vec{m} = 0$ ^{luka ravnine = normala u in točka O.}
 $\vec{m} = \text{vektor vrtilne količine} = \frac{\vec{L}(O, P_0)}{|\vec{L}(O, P_0)|}$

P_0 položaj v času t_0 :
 $P_0 = P(t=t_0)$

$P(t=t_0)-O) \cdot \vec{m} = 0$

$P(t=t_0)-O) \cdot (P(t=t_0)-O) \times m\dot{P}(t=t_0) = 0$

$(P(t)-O) \vec{L}(O, P_0) = (P(t)-O) \vec{L}(O, P(t)) = 0$

$\Rightarrow$ gibanje je ravno

polarní koordinátní systém

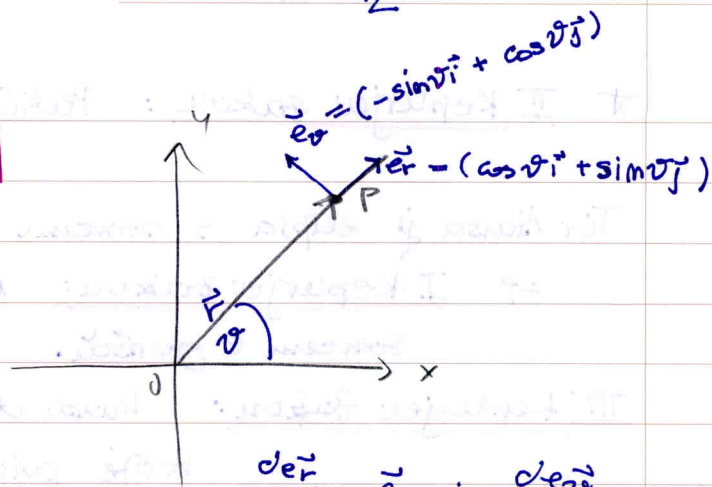
$$r, \vartheta; \quad \vec{r} = r\vec{e}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\vec{e}_r) = \dot{r}\vec{e}_r + r \frac{d\vec{e}_r}{d\vartheta} \dot{\vartheta}$$

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\vartheta}\vec{e}_\vartheta$$

radialní
hladnost

obdobná
hladnost



$$\frac{d\vec{e}_r}{d\vartheta} = \vec{e}_\vartheta; \quad \frac{d\vec{e}_\vartheta}{d\vartheta} = -\vec{e}_r$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\vec{e}_r + r\dot{\vartheta}\vec{e}_\vartheta) = \ddot{r}\vec{e}_r + \dot{r}\dot{\vartheta}\vec{e}_\vartheta + \dot{r}\dot{\vartheta}\vec{e}_\vartheta + r\ddot{\vartheta}\vec{e}_\vartheta + r\dot{\vartheta}(-\dot{\vartheta}\vec{e}_r) =$$

$$\vec{a} = (\ddot{r} - r\dot{\vartheta}^2)\vec{e}_r + (r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta})\vec{e}_\vartheta$$

radialní
normálek

obdobná
normálek

$$\vec{L} = (\vec{p} - \vec{O}) \times m\vec{v} = r\vec{e}_r \times m(\dot{r}\vec{e}_r + r\dot{\vartheta}\vec{e}_\vartheta) = m r^2 \dot{\vartheta} (\vec{e}_r \times \vec{e}_\vartheta)$$

$$\vec{L} = m r^2 \dot{\vartheta} \vec{k}; \quad L = m r^2 \dot{\vartheta}$$

Řávnímnožo gřbanje v poju centralne síle ima dve skalární konstanty gřbanja.

Ploščinška hladnost

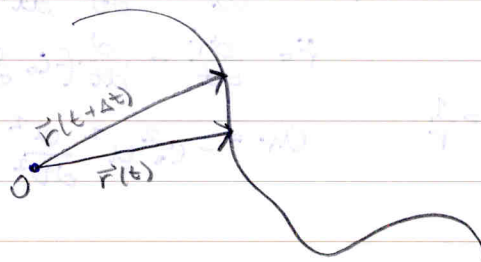
$$\Delta \vec{S} = \frac{1}{2} \vec{r}(t) \times (\vec{r}(t+\Delta t) - \vec{r}(t))$$

$$\frac{\Delta \vec{S}}{\Delta t} = \frac{1}{2} \vec{r}(t) \times \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$\downarrow \Delta t \rightarrow 0$

$$\frac{d\vec{S}}{dt}$$

$$= \frac{1}{2} \vec{r} \times \dot{\vec{r}}$$



$$\frac{d\vec{S}}{dt} = \frac{1}{2} \vec{r} \times \dot{\vec{r}} = \frac{1}{2} r^2 \dot{\vartheta} \vec{k}$$

$$\frac{d\vec{S}}{dt} = \frac{1}{2} C_0 \vec{k}$$

C_0 = "dvojha ploščinška hladnost"

$$L = m C_0$$

Tycho Brahe 1546 - 1601

Johannes Kepler 1571 - 1630

Isaac Newton 1642 - 1727

žemja se gřbje neman. po kružnici, souce m v sředstě.

★ II Keplerjev zakon: Ploščinska hitrost je konstantna
 $\rightarrow$ enaki $\Delta s \Rightarrow$ enaka Δt pe.

Tir Marsa je elipsa s soncem v gonšči elipse

$\Rightarrow$ I Keplerjev zakon: planeti se gibljejo po elipsah s soncem v gonšči.

III Keplerjev zakon: kvadrat periode je sorazmeren kubu veče polosi elipse.

Trditev: Če je ploščinska hitrost konstantna je obvodni nosp. enak nič.

Dokaz $C_0 = r^2 \dot{\varphi} \Rightarrow \frac{dC_0}{dt} = 2r\dot{r}\dot{\varphi} + r^2\ddot{\varphi} = 0 \Rightarrow r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 0$
 $a_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi}$
 $\Downarrow$
 $\underline{\underline{a_\varphi = 0}}$

Trditev: Če je tir elipsa, je radialni nospček ohratno sorazmeren kvadratu oddaljenosti.

Dokaz $a_\varphi = 0 \Rightarrow r^2 \dot{\varphi} = C_0$
 $a_r = \ddot{r} - r\dot{\varphi}^2$
 $r = r(\vartheta), \dot{r} = \frac{dr}{dt} = \frac{dr}{d\vartheta} \cdot \frac{d\vartheta}{dt} = \frac{dr}{d\vartheta} \left(\frac{C_0}{r^2} \right) = -C_0 \frac{d}{d\vartheta} \left(\frac{1}{r} \right)$
 $\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d}{dt} \left(-C_0 \frac{d}{d\vartheta} \left(\frac{1}{r} \right) \right) = -C_0 \frac{d^2}{d\vartheta^2} \left(\frac{1}{r} \right) \frac{C_0}{r^2} = -C_0^2 \frac{1}{r^2} \frac{d^2}{d\vartheta^2} \left(\frac{1}{r} \right)$
 $\mu = \frac{1}{r}$
 $a_r = -C_0^2 \mu^2 \frac{d^2}{d\vartheta^2} (\mu) - \frac{1}{\mu} (C_0 \mu^2)^2 = -C_0^2 \mu^2 \left(\frac{d^2 \mu}{d\vartheta^2} + \mu \right)$

Binetova enačba

Tir elipsa: v polarnem zapisu:

$$r = \frac{p}{1 + \varepsilon \cdot \cos \vartheta}$$

$\downarrow$
 ekscentricnost
 $\varepsilon = 0 \Rightarrow$ krožnica

$$\frac{du}{d\vartheta} = u' = -\frac{\varepsilon}{p} \sin \vartheta$$

$$u'' = -\frac{\varepsilon}{p} \cos \vartheta$$

$$a_r = -C_0^2 \frac{1}{r^2} \left(-\frac{\varepsilon}{p} \cos \vartheta + \frac{1}{p} + \frac{\varepsilon}{p} \cos \vartheta \right)$$

$$\underline{\underline{a_r = -\frac{C_0^2}{p} \cdot \frac{1}{r^2}}}$$

$$\vec{F}_g = -\frac{m_1 m_2}{r^2} \vec{e}_r$$

$$m_2 \vec{a} = \vec{F}_g$$

$$\underline{\underline{\vec{a} = -\frac{m_1}{r^2} \vec{e}_r}}$$

Trditelj Konstanta $\frac{C_0^2}{p}$ je enaka za vse paze sonce + planet.

Dokaz $T^2 = k a^3$ (= 3. Newton. zakon?)

Ploščinska hitrost je konst = $\frac{1}{2} C_0$.

$$\left(\frac{1}{2} C_0\right) \cdot T = \text{ploščina elipse}$$

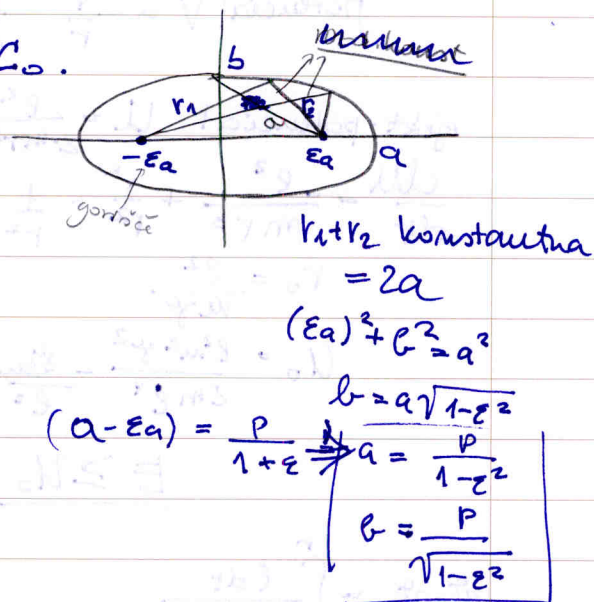
$\pi a b$

$$C_0 = \frac{2\pi a b}{T}$$

$$\frac{C_0^2}{p} = \frac{4\pi^2 a^2 b^2}{T^2 p} = \frac{4\pi^2 a^2 b^2}{k a^3 p} =$$

$$= \frac{4\pi^2}{k} \cdot \frac{b^2}{a p} = \frac{4\pi^2}{k} \frac{p^2}{(1-\varepsilon^2)p \cdot p} =$$

$$= \frac{4\pi^2}{k} \text{ konstanta}$$



$$C_0 = r^2 \dot{\vartheta}$$

$$E_0 = \frac{1}{2} m |\dot{\mathbf{r}}|^2 + V(r), \quad \mathbf{F} = -\text{grad } V$$

$$E_0 = \frac{1}{2} m |\dot{r}\mathbf{e}_r + r\dot{\vartheta}\mathbf{e}_\vartheta|^2 + V(r) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\vartheta}^2) + V(r) =$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \frac{C_0^2}{r^4} + V(r) =$$

$$\underline{E_0} = \frac{1}{2} m \dot{r}^2 + \frac{C_0^2}{2r^2} + V(r) = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} + V(r)$$

$$E_0 = \frac{1}{2} m \dot{r}^2 + U(r)$$

$$\Rightarrow r = r(t)$$

energ. enačba
za memotno
gibanje

$U(r)$ - efektivni
potencial

Gibanje v polju centralne sile smo reducirali na memotno gibanje koordinate r .

$$\dot{\vartheta} = \frac{C_0}{r^2} \Rightarrow \vartheta = \int_{t_0}^t \frac{C_0}{r^2(t)} dt + \vartheta_0$$

$$\frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = E_0 \Rightarrow t = t(r) \Rightarrow r = r(t) \quad \left. \begin{array}{l} \\ r^2 \dot{\vartheta} = C_0 \end{array} \right\} \vartheta = \vartheta(t)$$

ENACBA TIRA

$$\vartheta = \vartheta(r)$$

$$\frac{d\vartheta}{dr} = \frac{\partial \vartheta}{\partial t} \frac{dt}{dr} = \frac{C_0}{r^2} \frac{1}{\dot{r}} = \frac{C_0}{r^2} \frac{\text{sgn } \dot{r}}{\sqrt{\frac{2}{m}(E_0 - U(r))}}$$

$$d\vartheta = \pm \frac{C_0}{r^2} \frac{dr}{\sqrt{\frac{2}{m}(E_0 - U(r))}}$$

$$\vartheta - \vartheta_0 = \text{sgn } \dot{r} \int_{r_0}^r \frac{L dr}{\sqrt{2u(E_0 - U(r))}}$$

$$\left(\begin{array}{l} ar = -C_0^2 u^2 (u'' + u) \\ \text{max} = F \\ F = -\frac{1}{2} m \omega^2 u^2 \end{array} \right)$$

Primer gibanje u polju grav. sile (60 točka)

$$F = -\gamma m M \frac{1}{r^2} = -\gamma \frac{1}{r^2}$$

$$\text{potencijal } V = -\frac{\gamma}{r} ; \text{ grad } V = \frac{\partial V}{\partial r} \vec{e}_r = \frac{\gamma}{r^2} \vec{e}_r$$

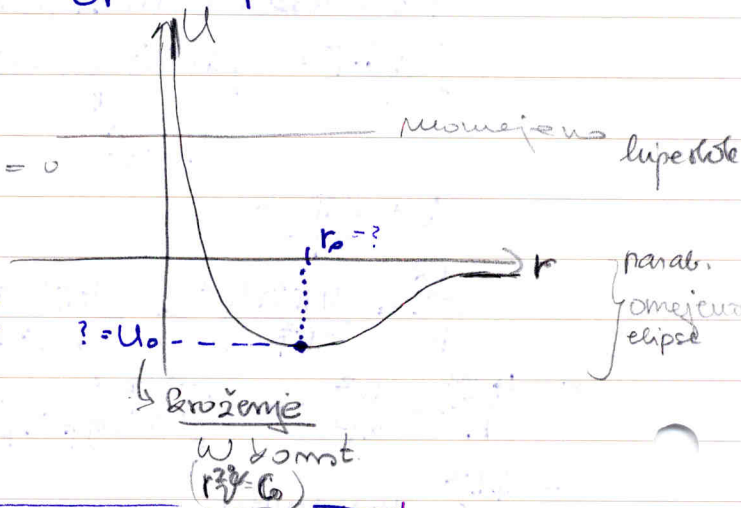
$$\text{efekt. potencijal } U = \frac{l^2}{2 \cdot m r^2} - \frac{\gamma}{r}$$

$$\frac{dU}{dr} = -\frac{l^2}{m r^3} + \frac{\gamma}{r^2} = \frac{1}{r^2} \left(\gamma - \frac{l^2}{m r} \right) = 0$$

$$r_0 = \frac{l^2}{m \gamma}$$

$$U_0 = \frac{l^2 m \gamma^2}{2 m l^4} - \frac{\gamma m \gamma}{l^2} = -\frac{m \gamma^2}{2 l^2}$$

$$E \geq U_0$$



$$\begin{aligned} \vartheta - \vartheta_0 &= \int_{r_0}^r \frac{l dr}{r^2 \sqrt{2m(E - \frac{l^2}{2mr^2} + \frac{\gamma}{r})}} \\ &= - \int_{\frac{l}{2m r_0}}^{\frac{l}{2m r}} \frac{du}{\sqrt{E - U^2 + \frac{\gamma^2 2m}{l^2} U}} \end{aligned}$$

$$\begin{aligned} \frac{l^2}{2m r^2} &= u^2 \\ r &= \frac{l}{2m u} \\ \frac{dr}{r^2} &= -\frac{l}{2m} \frac{1}{u^2} du \end{aligned}$$

$$= - \int_{u_0}^u \frac{du}{\sqrt{E + \frac{\gamma^2 2m}{4e^2} - \left(u - \frac{\gamma \sqrt{2m}}{2e}\right)^2}} = \arccos \left(\frac{u - \frac{\gamma \sqrt{2m}}{2e}}{\sqrt{E + \frac{\gamma^2 2m}{4e^2}}} \right)$$

$$= \frac{1}{\sqrt{E + \frac{\gamma^2 2m}{4e^2}}} \int_{u_0}^u \frac{du}{\sqrt{1 - \left(\frac{u - \frac{\gamma \sqrt{2m}}{2e}}{\sqrt{E + \frac{\gamma^2 2m}{4e^2}}} \right)^2}} = \arccos \left(\frac{u - \frac{\gamma \sqrt{2m}}{2e}}{\sqrt{E + \frac{\gamma^2 2m}{4e^2}}} \right) \Big|_{u_0}^u$$

$$\vartheta - \vartheta_0 = \arccos \frac{u - \frac{\gamma \sqrt{2m}}{2e}}{\sqrt{E + \frac{\gamma^2 2m}{4e^2}}} - \arccos \frac{\frac{l m \gamma}{2e} - \frac{\gamma \sqrt{2m}}{2e}}{\sqrt{E + \frac{\gamma^2 2m}{4e^2}}} = 0$$

$$\vartheta - \vartheta_0 \neq \frac{\pi}{2} = \arccos \frac{u - \frac{\gamma \sqrt{2m}}{2e}}{\sqrt{E + \frac{\gamma^2 2m}{4e^2}}}$$

$$\cos(\vartheta - \vartheta_0 + \frac{\pi}{2}) = \frac{u - \frac{\gamma \sqrt{2m}}{2e}}{\sqrt{E + \frac{\gamma^2 2m}{4e^2}}} \Rightarrow \frac{l}{2m r} = \frac{\gamma \sqrt{2m}}{2e} + \sqrt{E + \frac{\gamma^2 2m}{4e^2}} \cos(\vartheta - \vartheta_0 + \frac{\pi}{2})$$

$$\frac{1}{r} = \frac{\gamma \sqrt{2m}}{2e^2} + \sqrt{E + \frac{\gamma^2 2m}{4e^2}} \cos(\vartheta - \vartheta_0 + \frac{\pi}{2})$$

$$\frac{1}{r} = \frac{\gamma m}{e^2} + \frac{\sqrt{2m}}{e} \cdot \frac{\gamma \gamma}{\sqrt{2} e} \sqrt{1 + \frac{E e^2}{m \gamma^2}} \cos(\vartheta - \vartheta_0 + \frac{\pi}{2})$$

$$\text{elipsa: } r = \frac{p}{1 + \epsilon \cos \vartheta}$$

$$r = \frac{\gamma m}{e^2} \left(1 + \sqrt{1 + \frac{2E e^2}{m \gamma^2}} \cos(\vartheta) \right)$$

$$p = \frac{l^2}{\gamma m}$$

$$\epsilon = \sqrt{1 + \frac{2E e^2}{m \gamma^2}} ; \vartheta_0 = \frac{\pi}{2}$$

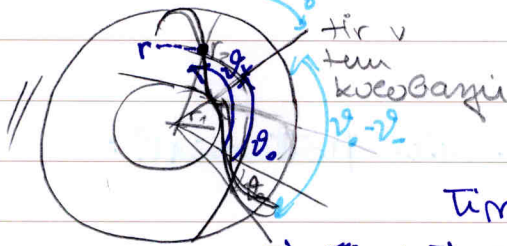
$$\Rightarrow r = \frac{p}{1 + \epsilon \cos \vartheta}$$

matrica elipse

1. kor: $E < 0 \Rightarrow \epsilon < 1 \rightarrow \text{elipsa}$

$$v - v_0 = \pm \int_{r_0}^r \frac{e}{r^2 \sqrt{2m(E - U(r))}} dr$$

OMEJENO GIBANJE: $r_1 \leq r \leq r_2$



r_1, r_2 apsidna radija
pericenter $\uparrow$ apocenter

Tir se dotika apsidalnih krožnic:

$$\vec{F} = \vec{F}(r) \cdot \vec{e}_r$$

$$r \cdot \vec{e}_\theta + \frac{dr}{d\theta} \vec{e}_r = \frac{d\vec{r}}{d\theta} \parallel \vec{e}_t \quad \leftarrow \text{pri } r = r_{1,2} \text{ je } \frac{dr}{d\theta} = 0$$

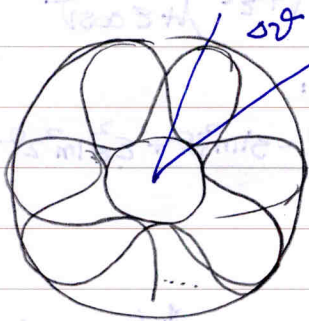
$$\left. \frac{dr}{d\theta} \right|_{r=r_{1,2}} = r_{1,2} \vec{e}_\theta; \quad \vec{e}_\theta = \vec{e}_t$$

Trditev Tir je simetričen glede na apsidni radij.

Dokaz $v_+ - v_0 = \int_{r_0}^{r_2} \frac{e}{r^2 \sqrt{E - U(r)}} dr$

$$v_- - v_0 = - \int_{r_0}^{r_1} \frac{e}{r^2 \sqrt{E - U(r)}} dr$$

$$\Rightarrow v_+ - v_0 = v_0 - v_-$$



Ali se tir kdaj zapre?

$$\Delta \varphi = \frac{e}{2m} \int_{r_1}^{r_2} \frac{dr}{r^2 \sqrt{E - U(r)}}$$

$$n \Delta \varphi = 2\pi \leftarrow \text{celenkrat okrog}$$

$$n \Delta \varphi = m 2\pi; \quad m, n \in \mathbb{N}$$

Trditev Tir je zaprt, če je $\frac{\Delta \varphi}{\pi} \in \mathbb{Q}$.

Trditev Če tir ni zaprt, je gosta množica v kolobanju.

Dokaz Arnold, knjiga (neka).

Za kateri potencial je vsak omejen tir zaprt?

Za gravitacijski potencial ($\frac{1}{r}$ od prej) je vsak omejeni tir zaprt.

Tudi Hookov potencial ima to lastnost:

$$\vec{F} = -k r \vec{e}_r, \quad V = \frac{1}{2} k r^2$$

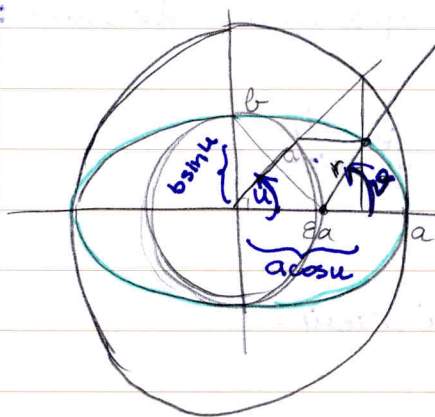
Izrek (Bertranchov izrek)

Če so vsi tri v oziroma vrtni količine in energije stabilnega krožnega tira $r=r_0$ zaprti in je potencial analitičen

v okolici r_0 je v tej okolici potencial gravitacijski ali Hookov.

$$U = \frac{p^2}{2mr^2} + V(r)$$

Določitev trajektorije za gravit. potencial:



Elipse

$$a^2 = b^2 + (ea)^2$$

$$b = a\sqrt{1-e^2}$$

$$a = \frac{p}{1-e^2}$$

$$r = \frac{p}{1+e\cos v}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = b \sin u$$

$$x = a \cos u \quad ; \quad u = \text{ekscentrična anomalija}$$

$$\begin{aligned} y &= a\sqrt{1-e^2} \sin u = \\ &= r \cdot \sin v = \\ &= \frac{p}{1+e\cos v} \cdot \sin v \end{aligned}$$

$$\sin u = \frac{p \cdot \sin v (1-e^2)}{p(1-e^2)(1+e\cos v)} = \sqrt{1-e^2} \frac{\sin v}{1+e\cos v}$$

Iščemo zvezo med u in v :

$$\cos^2 u = 1 - \sin^2 u = 1 - \frac{(1-e^2)\sin^2 v}{(1+e\cos v)^2} = \frac{1+2e\cos v + e^2\cos^2 v - \sin^2 v + e^2\sin^2 v}{(1+e\cos v)^2}$$

$$= \frac{\cos^2 v + 2e\cos v + e^2}{(1+e\cos v)^2}$$

$$\cos u = \frac{e + \cos v}{1 + e\cos v}$$

$$\operatorname{tg}^2 \frac{u}{2} = \frac{\sin^2 \frac{u}{2}}{\cos^2 \frac{u}{2}} = \frac{\frac{1}{2}(1-\cos u)}{\frac{1}{2}(1+\cos u)}$$

$$= \frac{1 - \frac{e + \cos v}{1 + e\cos v}}{1 + \frac{e + \cos v}{1 + e\cos v}}$$

$$\operatorname{tg}^2 \frac{u}{2} = \frac{1 + e\cos v - e - \cos v}{1 + e\cos v + e + \cos v} = \frac{(1-e)(1-\cos v)}{(1+e)(1+\cos v)} = \frac{1-e}{1+e} \operatorname{tg}^2 \frac{v}{2}$$

$$\operatorname{tg} \frac{u}{2} = \sqrt{\frac{1-e}{1+e}} \operatorname{tg} \frac{v}{2}$$

Iščemo enačbo za u . Rezitev je $u = u(t)$

Dobljeni enačbi pravimo Keplerjeva enačba.

Keplerjeva enačba

$$y = b \sin u$$

$$x = a \cos u$$

$$a\varepsilon + \xi = x$$

$$\xi = a(\cos u - \varepsilon)$$

$$\vec{r} = \xi \vec{i} + y \vec{j}$$

$$\frac{1}{2} l_0 = \left| \frac{d\vec{r}}{dt} \right| = \left| \frac{1}{2} \vec{r} \times \frac{d\vec{r}}{dt} \right| = \frac{1}{2} ab (1 - \varepsilon \cos u) \dot{u}$$

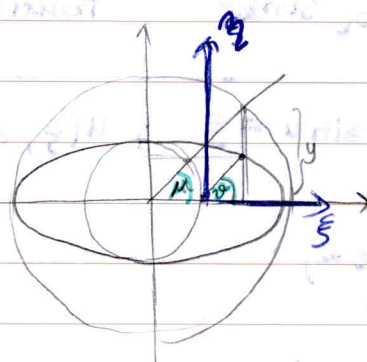
$$\frac{1}{2} l_0 = \frac{1}{2} ab (1 - \varepsilon \cos u) \dot{u} ; \dot{u} > 0$$

$$\frac{ab}{T}$$

$$\Rightarrow \frac{2\pi}{T} = (1 - \varepsilon \cos u) \dot{u} = \frac{d}{dt} (u - \varepsilon \sin u)$$

$$\frac{2\pi}{T} t = u - \varepsilon \sin u \quad (u(t=0) = 0)$$

$$\xi = \frac{2\pi b}{T} \rightarrow \boxed{\xi = u - \varepsilon \sin u}$$



$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{du} \dot{u} = (-a \sin u \vec{i} + b \cos u \vec{j}) \dot{u}$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = ((a \cos u - \varepsilon) \vec{i} + b \sin u \vec{j}) \times (-a \sin u \vec{i} + b \cos u \vec{j}) \dot{u} = (ab \cos^2 u \vec{k} + ab \sin^2 u \vec{k} - a\varepsilon b \cos u \vec{k}) \dot{u}$$

Trditev za $\forall \xi$ in $\varepsilon \in [0, 1)$ obstaja natanko ena rešitev Keplerjeve enačbe.

Dokaz

$$u - \varepsilon \sin u - \xi = 0$$

$$F(u, \xi) = 0$$

$$\boxed{\frac{\partial F}{\partial u} \neq 0 \Rightarrow u = u(\xi)}$$

$$\frac{\partial F}{\partial u} = 1 - \varepsilon \cos u > 0 \quad \checkmark$$

Trditev Funkcija $u(\xi) - \xi =$ liha funkcija (periodična) s periodo 2π .

Dokaz

$$u(\xi) - \varepsilon \sin u(\xi) = \xi$$

$$-u(\xi) + \varepsilon \sin u(\xi) = -\xi$$

$$(-u(\xi)) - (\varepsilon \sin(-u(\xi))) = -\xi \quad \Rightarrow u(-\xi) = -u(\xi) \quad (\text{lihost})$$

$$(u(\xi) + 2\pi) - \varepsilon \sin(u(\xi) + 2\pi) = \xi + 2\pi$$

$$u(\xi + 2\pi) = u(\xi) + 2\pi$$

$\Rightarrow$ vstavimo in pride isto $\Rightarrow$ periodično

$$u(\xi) - \xi = \sum_{k=1}^{\infty} b_k \sin k\xi \quad \text{Dokaz 2} \quad \text{Fourierova vrsta} \quad b_k = b_k(\varepsilon)$$

ali: $u = \varepsilon \sin u = \xi$; $u(\xi, \varepsilon) = \sum_{k=0}^{\infty} \frac{1}{k!} C_k(\xi) \cdot \varepsilon^k$

$$C_k(\xi) = \frac{d^k u}{d\varepsilon^k}(\varepsilon=0)$$

$$C_0 = u(\varepsilon=0) = \xi$$

$$C_1 = \frac{du}{d\varepsilon}(\varepsilon=0) \Rightarrow \frac{du}{d\varepsilon} - \sin u - \varepsilon \cos u \frac{du}{d\varepsilon} = 0$$

$$\frac{du}{d\varepsilon}(\varepsilon=0) - \sin(u(\varepsilon=0)) = 0 \Rightarrow \frac{du}{d\varepsilon}(\varepsilon=0) = \sin \xi$$

$$C_2 = \frac{d^2 u}{d\varepsilon^2} \Rightarrow \frac{d^2 u}{d\varepsilon^2} - 2 \cos u \frac{du}{d\varepsilon} - \varepsilon \frac{d}{d\varepsilon} \left(\cos u \frac{du}{d\varepsilon} \right) = 0$$

$$C_2 \Rightarrow \frac{d^2 u}{d\varepsilon^2} = 2 \cos(\xi \cdot \sin \xi) = \frac{d}{d\xi} \sin^2 \xi$$

induktivno

$$C_k = \frac{d^{k-1}}{d\xi^{k-1}} \sin^k \xi$$

$$u(\xi, \varepsilon) = \xi + \varepsilon \sin \xi + \frac{1}{2} \varepsilon^2 \sin 2\xi + O(\varepsilon^3)$$

Konvergenca? Znamo je, da vrsta konvergira za $0 \leq \varepsilon \leq 0.6627...$

Gibanje v relativnem koord. sistemu

RKS

$$P' = P_0' + Q(P - P_0)$$

$$t' = t$$

2 koord. sistema

$$\begin{cases} \varphi(t, P) \\ \varphi'(t', P') \end{cases}$$

Def KS $\varphi(t, P)$ se gibeje glede na KS' $\varphi'(t', P')$, če obstaja trojica (P_0, P_0', Q) , kjer je P_0 fiksna točka $\in \mathbb{E}$

$$P_0' : t \rightarrow P_0'(t)$$

$$Q : t \rightarrow Q(t) \in SO(3),$$

da je $t' = t$ in $P' = P_0'(t) + Q(t)(P - P_0)$

translacijski del

postaneta funkciji časa

rotacijski del

center gibanja

Trditev Rotacijski del gibanja je neodvisen od izbrane centra gibanja.

Dokaz $\exists (P_0, P_0', Q) \rightarrow P' = P_0'(t) + Q(t)(P - P_0) \quad \forall P$
in $\exists (\hat{P}_0, \hat{P}_0', \hat{Q}) \rightarrow P' = \hat{P}_0'(t) + \hat{Q}(t)(P - \hat{P}_0)$
 $P = P_0 \Rightarrow P_0'(t) = P' = \hat{P}_0'(t) + \hat{Q}(t)(P - \hat{P}_0)$
 $\Rightarrow \hat{P}_0'(t) = P_0'(t) - \hat{Q}(t)(P_0 - \hat{P}_0)$
 $\Rightarrow P' = P_0'(t) - \hat{Q}(t)(P_0 - \hat{P}_0) + \hat{Q}(t)(P - \hat{P}_0) =$
 $= P_0'(t) + \hat{Q}(t)((P - \hat{P}_0) - (P_0 - \hat{P}_0)) \Rightarrow$
 $\Rightarrow \frac{P' = P_0'(t) + \hat{Q}(t)(P - P_0)}{\text{in } P' = P_0'(t) + Q(t)(P - P_0)} \} \Rightarrow \underline{Q(t) = \hat{Q}(t)}$

RKS $\psi(t, P)$ relativni KS

AKS $\psi'(t', P')$ absolutni KS

$$P' = P_0'(t) + Q(t)(P - P_0)$$

povezuje oba opazovalca

$$\dot{Q}(t) = ?$$

$$Q \in SO(3)$$

$$Q Q^T = Q^T Q = I \quad \forall t$$

$$\frac{d}{dt}(Q Q^T) = 0$$

$$\frac{dQ}{dt} Q^T + Q \frac{dQ^T}{dt} = 0$$

$$\dot{Q} Q^T + Q \dot{Q}^T = 0 \quad \text{"} \dot{Q} Q^T \text{"}$$

$$\dot{Q}^T Q + Q^T \dot{Q} = 0$$

$$(\dot{Q}^T Q)^T = 0$$

$$\dot{Q} Q^T + (\dot{Q} Q^T)^T = 0$$

$$\dot{Q}^T Q + (\dot{Q}^T Q)^T = 0$$

$W := \dot{Q}^T Q$ je poševno simetr. tenzor.

$$\text{na diag} = 0$$

$$(\dot{Q})^T \stackrel{?}{=} (Q^T)^{\cdot}$$

$$Q \vec{a} \cdot \vec{b} = \vec{a} Q^T \vec{b}, \quad \vec{a}, \vec{b} \text{ konst}$$

$$\dot{Q} \vec{a} \cdot \vec{b} = \vec{a} (\dot{Q}^T)^{\cdot} \vec{b}$$

$$\vec{a} (\dot{Q})^T \vec{b} = \vec{a} (\dot{Q}^T)^{\cdot} \vec{b} \quad \forall \vec{a}$$

$$\Rightarrow (\dot{Q})^T \vec{b} = (\dot{Q}^T)^{\cdot} \vec{b} \quad \forall \vec{b} \quad \left. \vphantom{\vec{b}} \right\} \dot{Q}^T = (\dot{Q}^T)^{\cdot} = \underline{\underline{\dot{Q}^T}}$$

$$W + W^T = 0$$

$$W^T = -W$$

poševno
simetrični tenzor

Izrek Naj bo W poš. sim. tenzor med trirazsežnim vekt. prost. Potem obstaja vektor $\vec{w}$ tako, da je $W\vec{a} = \vec{w} \times \vec{a}$.

Vektorji $\vec{w}$ maximo osni vektor poš. sim. tenzorja W .

Dokaz $W\vec{p} = \lambda\vec{p}$, $\vec{p}W\vec{p} = \vec{p}\lambda\vec{p} = \lambda|\vec{p}|^2$; $W^T\vec{p}\vec{p} = -W\vec{p}\vec{p} = -\lambda|\vec{p}|^2$

$$\vec{p}W\vec{p} = W^T\vec{p}\vec{p}$$

$$\Rightarrow \lambda = 0 \quad \text{edina rešitev}$$

Alketa

$\omega \neq 0 \Rightarrow$ ima eno l. vrstu $\Omega = 0$.

$\omega \vec{p} = 0, |\vec{p}| = 1$; $\vec{p}, \vec{q}, \vec{r}$ ortonomirana baza
 $\omega \vec{q} = (\underbrace{\vec{p} \cdot \omega \vec{q}}_{\substack{\text{skal.} \\ \text{proizvoda} \\ = 0}}) \vec{p} + (\underbrace{\vec{q} \cdot \omega \vec{q}}_{= 0}) \vec{q} + (\vec{r} \cdot \omega \vec{q}) \vec{r} = \underline{(\omega \vec{q} \cdot \vec{r}) \vec{r}}$

$\omega \vec{r} = (\omega \vec{r} \cdot \vec{q}) \vec{q}$
skal. smer

$$\underline{\underline{\omega \vec{a} = \vec{\omega} \times \vec{a}}}$$

$$\underline{\underline{\vec{\omega} = (\omega \vec{q} \cdot \vec{r}) \vec{p}}}$$

$$\omega \vec{a} = \omega ((\vec{a} \cdot \vec{p}) \vec{p} + (\vec{a} \cdot \vec{q}) \vec{q} + (\vec{a} \cdot \vec{r}) \vec{r}) = (\vec{a} \cdot \vec{q}) \omega \vec{q} + (\vec{a} \cdot \vec{r}) \omega \vec{r} =$$

$$= (\vec{a} \cdot \vec{q}) (\omega \vec{q} \cdot \vec{r}) \vec{r} + (\vec{a} \cdot \vec{r}) (\omega \vec{r} \cdot \vec{q}) \vec{q}$$

$$\vec{\omega} \times \vec{a} = (\omega \vec{q} \cdot \vec{r}) \vec{p} \times ((\vec{a} \cdot \vec{p}) \vec{p} + (\vec{a} \cdot \vec{q}) \vec{q} + (\vec{a} \cdot \vec{r}) \vec{r}) =$$

$$= (\omega \vec{q} \cdot \vec{r}) ((\vec{a} \cdot \vec{q}) \vec{p} \times \vec{q} + (\vec{a} \cdot \vec{r}) \vec{p} \times \vec{r}) =$$

$$\Rightarrow \underline{\underline{\omega \vec{a} = \vec{\omega} \times \vec{a}}}$$

$$\begin{cases} \vec{p} \times \vec{q} = \vec{r} \\ \vec{q} \times \vec{r} = \vec{p} \\ \vec{r} \times \vec{p} = \vec{q} \end{cases}$$

Trditaj $\omega_{ij} = -\epsilon_{ijk} \omega_k =$
 $= -\sum_{k=1}^3 \epsilon_{ijk} \omega_k$

$$\epsilon_{ijk} = \begin{cases} 1 & ; \text{ ijk soda permutacija števil 1,2,3} \\ -1 & ; \text{ liha} \\ 0 & ; \text{ sicer} \end{cases}$$

Dokaz $\omega_{ij} = \vec{e}_i \cdot \omega \vec{e}_j = \vec{e}_i \cdot (\vec{\omega} \times \vec{e}_j) = \vec{\omega} \cdot (\vec{e}_j \times \vec{e}_i)$

$$\underline{\underline{\omega_{ij} = \epsilon_{jik} \vec{e}_k \cdot \vec{\omega} = \epsilon_{jik} \omega_k = -\epsilon_{ijk} \omega_k \quad \checkmark}}$$

$$\vec{P}' = \vec{P}_0'(t) + \vec{Q}(t)(\vec{P} - \vec{P}_0)$$

$$\vec{V} = \frac{d\vec{P}}{dt} = \dot{\vec{P}}_0'(t) + \dot{\vec{Q}}(t)(\vec{P} - \vec{P}_0)$$

$$\dot{\vec{Q}} = ?$$

$$\vec{Q}^T \vec{Q} = I \Rightarrow \vec{Q}^T \dot{\vec{Q}} \text{ je p. sim. tenzor} \stackrel{n=3}{\Rightarrow} \exists \vec{\omega} \quad (\vec{Q}^T \dot{\vec{Q}}) \vec{a} = \vec{\omega} \times \vec{a}$$

Definicija Vektorja rotacije $\vec{\omega}$ je osni vektor tenzorja $\vec{Q}^T \dot{\vec{Q}}$.
 Vektorju $\vec{\omega}' = \vec{Q} \vec{\omega}$ pravimo vektor kotne hitrosti.

Trditaj

Vektor kotne hitrosti rotacije okrog stalne osi $\vec{e}$ je $\vec{\omega}' = \dot{\varphi} \vec{e}$.

Dokaz

$$Q = R(\vec{e}, \varphi) \quad \frac{d\vec{e}}{dt} = \vec{0}$$

 $\vec{e}$ je os rotacije $\Leftrightarrow Q\vec{e} = \vec{e}$ ($\vec{e}$ lastni vektor matrike Q)

$$\dot{Q}\vec{e} = \vec{0}$$

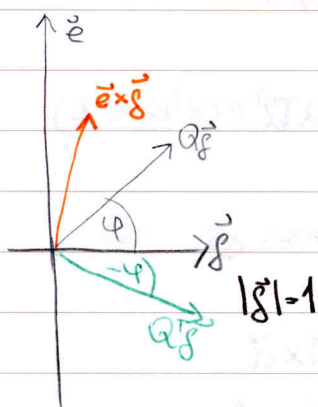
$$\vec{0} = Q^T \dot{Q} \vec{e} = \vec{\omega} \times \vec{e} \Rightarrow \vec{\omega} \parallel \vec{e}$$

$$\vec{\omega} = \omega \vec{e}$$

$$\omega = \dot{\varphi}$$

$$\vec{\omega}' = Q\vec{\omega} = Q(\omega \vec{e}) = \omega(Q\vec{e}) = \omega \vec{e} = \vec{\omega}$$

 vektor kotne
 hitrosti
 rot. gibanja

 Torej, če je rotacija okrog stalne osi, je $\vec{\omega}' = \vec{\omega}$
 Velja celo $\Leftrightarrow$.


$$Q\vec{f} \cdot \vec{f} = \cos \varphi$$

$$\dot{Q}\vec{f} \cdot \vec{f} = -\dot{\varphi} \sin \varphi$$

$$Q^T \dot{Q} \vec{f} \cdot Q\vec{f} = -\dot{\varphi} \sin \varphi$$

$$(\vec{\omega}' \times \vec{f}) \cdot Q\vec{f}$$

$$-\dot{\varphi} \sin \varphi = \omega (\vec{e} \times \vec{f}) \cdot Q\vec{f} = \omega \cos(\frac{\pi}{2} + \varphi) = -\omega \sin \varphi$$

$$Q = R(\vec{e}, \varphi) \Rightarrow Q^T = R(\vec{e}, -\varphi) \Rightarrow \underline{\underline{\omega = \dot{\varphi}}}$$

$$\text{Rotacija } Q(\vec{a} \times \vec{b}) \stackrel{?}{=} (Q\vec{a}) \times (Q\vec{b})$$

Trditev

 $\vec{a}, \vec{b}, \vec{c}$ poljubni lin. neodvisni vektorji.

$$(\vec{a} \cdot \vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$\det A = \frac{(\vec{A}\vec{a}, \vec{A}\vec{b}, \vec{A}\vec{c})}{(\vec{a}, \vec{b}, \vec{c})}$$

 merilni
 produkt

 $\|A\|$ (= treh invarianta)

$$\left(\frac{1}{(\vec{a}, \vec{b}, \vec{c})} (\vec{A}\vec{a}, \vec{A}\vec{b}, \vec{A}\vec{c}) + (\vec{a}, \vec{A}\vec{b}, \vec{c}) + (\vec{a}, \vec{b}, \vec{A}\vec{c}) \right) = I_A \quad \text{prva invarianta}$$

Dokaz

 $[\vec{a}, \vec{b}, \vec{c}]$ = matrika s stolpci $\vec{a}, \vec{b}, \vec{c}$

$$\det[\vec{a}, \vec{b}, \vec{c}] = (\vec{a}, \vec{b}, \vec{c}) \quad (\text{merilni produkt})$$

$$[\vec{A}\vec{a}, \vec{A}\vec{b}, \vec{A}\vec{c}] = A[\vec{a}, \vec{b}, \vec{c}]$$

$$\det[\vec{A}\vec{a}, \vec{A}\vec{b}, \vec{A}\vec{c}] = (\vec{A}\vec{a}, \vec{A}\vec{b}, \vec{A}\vec{c})$$

$$\det A \cdot \det[\vec{a}, \vec{b}, \vec{c}] = (\det A)(\vec{a}, \vec{b}, \vec{c}) \quad \checkmark$$

Trditev

 A = linearna preslikava $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$. Potem je

$$(\vec{A}\vec{a}) \times (\vec{A}\vec{b}) = A^* (\vec{a} \times \vec{b})$$

$$A^* = \text{konjugirana} \Rightarrow \boxed{A^{-1} = \frac{1}{\det A} (A^*)^T}$$

Dokaz $\det A (\vec{a}, \vec{b}, \vec{c}) = \det A (\vec{a} \times \vec{b}) \cdot \vec{c}$
 $(A\vec{a}, A\vec{b}, A\vec{c}) = ((A\vec{a}) \times (A\vec{b})) \cdot A\vec{c} = \det(A) (\vec{a} \times \vec{b}) \cdot A^{-1} A \vec{c} =$
 $= \underbrace{(\det A)}_{(A^{-1})^T} (\vec{a} \times \vec{b}) \cdot A\vec{c} \quad \forall \vec{c}$
 $\Rightarrow (\vec{a} \times \vec{b}) \cdot A\vec{c} = \det A \cdot A^{-T} (\vec{a} \times \vec{b}) \cdot \vec{c} = A^* (\vec{a} \times \vec{b}) \cdot \vec{c}$

Teorema $Q \in SO(3)$ (čije je Q rotacija) $\Rightarrow Q(\vec{a} \times \vec{b}) = (Q\vec{a}) \times (Q\vec{b})$

Dokaz $(Q\vec{a}) \times (Q\vec{b}) = Q^* (\vec{a} \times \vec{b})$ vemo $= Q(\vec{a} \times \vec{b})$
 $Q^* = (\det Q) (Q^{-T}) = Q$
 (rotacija ima $\det 1$); $Q^{-1} = Q^T$ ($Q Q^T = Q^T Q = I$)

Teorema Naj bo $\vec{\omega}$ vektor rotacije za Q . Potem je $-\vec{\omega}$ vektor kotne hitrosti rotacije Q^T .

Dokaz $\vec{\omega}$ rotacijski vektor za $Q \Rightarrow Q^T \dot{Q} \vec{a} = \vec{\omega} \times \vec{a}$
 $\vec{\eta}$ rotacijski vektor za $Q^T \Rightarrow (Q^T)^T (\dot{Q}^T)^T = \vec{\eta} \times \vec{a}$
 $(\dot{Q}^T)^T$

$$\frac{Q \dot{Q}^T \vec{a}}{\parallel} = \vec{\eta} \times \vec{a} ; \text{ vemo } (Q \dot{Q}^T)^T = -\dot{Q} Q^T$$

$$\dot{Q} Q^T \vec{a} = -\vec{\eta} \times \vec{a} \quad \vec{a} \dot{Q}^T$$

$$\dot{Q} Q^T (Q \vec{a}) = -\vec{\eta} \times Q \vec{a}$$

$$\Rightarrow \dot{Q} \vec{a} = -\vec{\eta} \times Q \vec{a}$$

$$Q^T \dot{Q} \vec{a} = Q^T (-\vec{\eta} \times Q \vec{a}) = -Q^T \vec{\eta} \times \vec{a}$$

$$(Q^T \dot{Q} \vec{a} = -(\dot{Q}^T Q) \times \vec{a}) \Rightarrow \vec{\omega} = -Q^T \vec{\eta}$$

Vektor kotne hitrosti za Q^T je $Q^T \vec{\eta} = -\vec{\omega}$.

$$R(\vec{e}, \varphi) \vec{r} = \vec{r} = ?$$

Teorema $R(\vec{e}, \varphi) \vec{r} = \cos \varphi \vec{r} + (\vec{r} \cdot \vec{e}) (1 - \cos \varphi) \vec{e} + (\vec{e} \times \vec{r}) \sin \varphi$

Dokaz

$\vec{r} \neq \vec{e} \rightarrow$

$\vec{r} = \lambda \vec{e}$
 če $\vec{r} \parallel \vec{e} \Rightarrow \cos \varphi \vec{r} + (\vec{r} \cdot \vec{e}) \vec{e} (1 - \cos \varphi) + (\vec{e} \times \vec{r}) \sin \varphi =$
 $= \cos \varphi (\lambda \vec{e}) + \lambda \vec{e} (1 - \cos \varphi) = \lambda \vec{e} = \vec{r}$
 $\vec{e}, \vec{r}, \vec{e} \times \vec{r}$ lin neodvisni

$$\vec{r}' = \alpha \vec{r} + \beta \vec{e} + \gamma \vec{e} \times \vec{r}$$

$$1. |\vec{r}'| = |\vec{r}|$$

$$2. \vec{r}' \cdot \vec{e} = \vec{r} \cdot \vec{e}$$

$$3. (\vec{e} \times \vec{r})(\vec{e} \times \vec{r}') = |\vec{e} \times \vec{r}|^2 \cos \varphi$$

} imać α, β, γ

$$\circ \vec{r} \cdot \vec{e} = \vec{r}' \cdot \vec{e} = \alpha (\vec{r} \cdot \vec{e}) + \beta \Rightarrow \beta = (\vec{r} \cdot \vec{e})(1 - \alpha) \quad \checkmark$$

$$\circ \vec{r}'^2 = \vec{r}' \cdot \vec{r}' = \vec{r}' \cdot \vec{r} = \alpha^2 r^2 + \beta^2 + \gamma^2 |\vec{e} \times \vec{r}|^2 + 2\alpha\beta \vec{r} \cdot \vec{e} =$$

$$r^2 = \alpha^2 r^2 + (\vec{r} \cdot \vec{e})^2 (1 - \alpha)^2 + \gamma^2 (r^2 - (\vec{e} \cdot \vec{r})^2) + 2\alpha(\vec{r} \cdot \vec{e})^2 (1 - \alpha) \Rightarrow$$

$$0 = r^2 (\alpha^2 + \gamma^2 - 1) + (\vec{r} \cdot \vec{e})^2 ((1 - \alpha)^2 - \gamma^2 + 2\alpha(1 - \alpha)) =$$

$$= r^2 (\alpha^2 + \gamma^2 - 1) + (\vec{r} \cdot \vec{e})^2 (1 - \gamma^2 - \alpha^2)$$

$$\left[\begin{aligned} & (\vec{e} \times \vec{r}) (\vec{e} \times \vec{r}') = (\vec{e} \cdot \vec{e}) (\vec{r} \cdot \vec{r}') - (\vec{e} \cdot \vec{r}') (\vec{e} \cdot \vec{r}) = \\ & = r^2 - (\vec{e} \cdot \vec{r})^2 \end{aligned} \right]$$

$$\Rightarrow \alpha^2 + \gamma^2 - 1 = 0$$

$$\alpha^2 + \gamma^2 = 1$$

$$\circ \vec{e} \times \vec{r}' = \alpha (\vec{e} \times \vec{r}) + \gamma \vec{e} \times (\vec{e} \times \vec{r})$$

$$(\vec{e} \times \vec{r}') (\vec{e} \times \vec{r}') = \alpha (\vec{e} \times \vec{r}) (\vec{e} \times \vec{r}) + \gamma (\vec{e} \times \vec{r}) \cdot (\vec{e} \times (\vec{e} \times \vec{r})) = \alpha |\vec{e} \times \vec{r}|^2$$

$$\cos^2 \varphi + \gamma^2 = 1$$

$$\gamma = \sin \varphi \quad \checkmark$$

$$\alpha = \cos \varphi \quad \checkmark$$

Definicija Tenzorski produkt dvuh vektorjev $\vec{u}$ in $\vec{v}$ je lin. preslikava, dana z:

$$(\vec{u} \otimes \vec{v}): \vec{a} \mapsto (\vec{u} \otimes \vec{v}) \vec{a} = (\vec{v} \cdot \vec{a}) \vec{u}$$

$$Q = \mathbb{L}(\vec{e}, \varphi) = (\cos \varphi I + (1 - \cos \varphi) \vec{e} \otimes \vec{e} + \sin \varphi W(\vec{e}))$$

$W(\vec{e})$ je poš. sim. tenzor z oskim vektorjem $\vec{e}$.

$$Q \in SO(3)$$

$$\text{Trditev } \det(\vec{u} \otimes \vec{v}) = \vec{u} \cdot \vec{v}$$

$$Q = R(\vec{e}, \varphi)$$

$$Q\vec{e} = \vec{e} \text{ (1. vektor)}$$

Dokaz

$\vec{e}_i$ orton. baza

$$\begin{aligned} \mathcal{L}(\vec{u} \otimes \vec{v}) &= \sum_{i=1}^3 \vec{e}_i \cdot (\vec{u} \otimes \vec{v}) \vec{e}_i = \vec{e}_i \cdot ((\vec{v} \cdot \vec{e}_i) \vec{u}) = \\ &= (\vec{e}_i \cdot \vec{u}) (\vec{e}_i \cdot \vec{v}) = \sum_{i=1}^3 u_i v_i = \vec{u} \cdot \vec{v}. \end{aligned}$$

Trditiv

$$\begin{aligned} \text{sl}(R(\vec{e}, \varphi)) &= \text{sl}(\cos \varphi I + (1 - \cos \varphi) \vec{e} \otimes \vec{e} + \sin \varphi W(\vec{e})) = \\ &= \text{sl}(\cos \varphi I + \cos \varphi \text{sl}(I) + (1 - \cos \varphi) \text{sl}(\vec{e} \otimes \vec{e})) + \sin \varphi \text{sl}(W) = \\ &= 3 \cos \varphi + (1 - \cos \varphi) + 0 = 2 \cos \varphi + 1 \end{aligned}$$

$\vec{e} \cdot \vec{e} = 1$

za $n=3$: $\Rightarrow \cos \varphi = \frac{1}{2} (\text{sl}(Q) - 1)$

~~za $n=2$: $2 \cos \varphi = \text{sl}(I)$~~

~~$\cos \varphi + 1 = \text{sl}(Q)$~~

~~$\cos \varphi = \text{sl}(Q) - 1$~~

Σ ravlina, $\Sigma \perp \vec{e}$
 $Q|_{\Sigma} = \cos I|_{\Sigma} + \sin \varphi W(\vec{e})|_{\Sigma}$

$\vec{r} \in \Sigma$: $(\vec{e} \otimes \vec{e})\vec{r} = (\vec{e}\vec{r})\vec{e} = 0$
 $\vec{r} \cdot \vec{e} = 0$

$\text{sl}(Q)|_{\Sigma} = 2 \cos \varphi$

$n=2$: $\cos \varphi = \frac{1}{2} \text{sl}(Q)$

$R(\vec{e}, \varphi)|_{(\vec{e}_j)} = Q = \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} \Rightarrow \text{sl} Q = 2 \cos \varphi$

$R(\vec{e}, \varphi) = \begin{pmatrix} \cos & -\sin & 0 \\ \sin & \cos & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vec{\omega}' = ?$

$\vec{a}' = Q \vec{a} = \vec{\omega} \times \vec{a}$

$\vec{\omega}' = Q \vec{\omega}$

Predstavitev rotacije s kvaternioni H .

(= obseg) mekomutativ

$\vec{a}, \vec{b} \in H$

$a = \alpha + \vec{a}$ (sk. del) vektorski del

$b = \beta + \vec{b}$

$a + b = (\alpha + \beta) + (\vec{a} + \vec{b})$

$a \cdot b = (\alpha \cdot \beta - \vec{a} \cdot \vec{b}) + (\alpha \vec{b} + \beta \vec{a} + \vec{a} \times \vec{b})$ (sk. del) vekt. del

konjug. $a^* = \alpha - \vec{a}$; $a \cdot a^* = \alpha^2 + |\vec{a}|^2 = |a|^2$

$q \in H$
 $q = \cos \frac{\varphi}{2} + \vec{e} \sin \frac{\varphi}{2}$, $|\vec{e}| = 1$

Trditiv $R = R(\vec{e}, \varphi)$

rotacija $R \vec{r} = q \vec{r} q^*$ kvat. produkt

Dokaz $q\vec{r} = (\cos \frac{\varphi}{2} + \vec{e} \sin \frac{\varphi}{2}) \vec{r} = (-\vec{e}\vec{r}) \sin \frac{\varphi}{2} + (\cos \frac{\varphi}{2} \vec{r} + \sin \frac{\varphi}{2} \vec{e} \times \vec{r})$
 $q\vec{r}q^* = ((\vec{e}\vec{r}) \sin \frac{\varphi}{2} + (\cos \frac{\varphi}{2} \vec{r} + \sin \frac{\varphi}{2} \vec{e} \times \vec{r})) (\cos \frac{\varphi}{2} - \vec{e} \sin \frac{\varphi}{2}) =$
 $= \underbrace{(-\vec{e}\vec{r}) \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} - (-\vec{r}\vec{e}) \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} + 0}_{\text{skal. del}} + \underbrace{(\vec{e}\vec{r})\vec{e} \sin^2 \frac{\varphi}{2}}_{\text{vekt. del}} +$
 $+ \underbrace{\cos^2 \frac{\varphi}{2} \vec{r}}_{\cos \varphi} + \underbrace{\cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \vec{e} \times \vec{r}}_{2 \sin^2 \frac{\varphi}{2} = 1 - \cos \varphi} + \underbrace{\cos \frac{\varphi}{2} \sin \frac{\varphi}{2} (\vec{r} \times (-\vec{e}))}_{\sin \varphi} - \underbrace{\sin^2 \frac{\varphi}{2} (\vec{e} \times \vec{r}) \vec{e}}_{\vec{r} - (\vec{r}\vec{e})\vec{e}}$
 $= (\cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2}) \vec{r} + (\vec{e}\vec{r})(\sin^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2}) \vec{e} + (2 \cos \frac{\varphi}{2} \sin \frac{\varphi}{2}) \vec{e} \times \vec{r} =$
 $= R(\vec{e}, \varphi) \vec{r}$

$$a = \alpha + \vec{a}$$

$$|a| = 1 \Rightarrow \alpha^2 + |\vec{a}|^2 = 1$$

$$\alpha = \cos \psi, \vec{a} = \sin \psi \vec{e}$$

Def Kvaterniona q_1, q_2 sta ekvivalentna, če $q_1 \vec{r} q_1^* = q_2 \vec{r} q_2^*$ za $\forall \vec{r}$.
 v relaciji $\mathcal{R}$

Trditev $\mathcal{R}$ je ekvivalenčna relacija nad prostorom kvaternionov.

Dokaz $\rightarrow q \mathcal{R} q$ očitno ($q \vec{r} q^* = q \vec{r} q^*$)
 $\rightarrow q_1 \mathcal{R} q_2 \Rightarrow q_2 \mathcal{R} q_1$ očitno ($q_1 \vec{r} q_1^* = q_2 \vec{r} q_2^* \Rightarrow q_2 \vec{r} q_2^* = q_1 \vec{r} q_1^*$)
 $\rightarrow q_1 \mathcal{R} q_2$ in $q_2 \mathcal{R} q_3 \Rightarrow q_1 \mathcal{R} q_3$ ✓

$$j: SO(3) \longrightarrow \mathbb{H}_1 \text{ (euklidi kvat.)}$$

$$j: R(\vec{e}, \varphi) \longmapsto \cos \frac{\varphi}{2} + \vec{e} \sin \frac{\varphi}{2}$$

Prelikava $j: SO(3) \longrightarrow \mathbb{H}_1 / \mathcal{R}$ je izomorfizem grup.
 rotacji R_1, R_2 , $j(R_2 \circ R_1) = j(R_2) \cdot j(R_1)$

$$q_1 = j(R_1) \quad , \quad q_2 = j(R_2)$$

$$\begin{aligned} \forall \vec{r}: (q_2 q_1) \vec{r} (q_2 q_1)^* &= (q_2 q_1) \vec{r} (q_1^* q_2^*) \stackrel{\text{asoc}}{=} q_2 (q_1 \vec{r} q_1^*) q_2^* = R_2(R_1 \vec{r}) = \\ &= (R_2 \circ R_1) \vec{r} = j(R_2 R_1) \vec{r} j(R_2 R_1)^* = R_1 \vec{r} \end{aligned}$$

$\Rightarrow q_2 q_1$ in $j(R_2 R_1)$ sta ekvivalentna (torej enaka) v rel. $\mathcal{R}$

$$\Leftrightarrow j(R_2) j(R_1) = j(R_2 R_1)$$

Izrek Rotacijski vektor rotacije $R(\vec{e}, \varphi)$ je $\vec{\omega} = \dot{\varphi} \vec{e} + \sin \varphi \ddot{\varphi} \vec{e} - (1 - \cos \varphi) \ddot{\varphi} \vec{e} \times \vec{e}$.

Vektor kotne hitrosti pa je $\vec{\omega}' = \dot{\varphi} \vec{e} + \sin \varphi \ddot{\varphi} \vec{e} + (1 - \cos \varphi) \ddot{\varphi} \vec{e} \times \vec{e}$

Dokaz $Q^T Q \vec{r} = \vec{\omega} \times \vec{r}$

$$\frac{d}{dt} (Q^T \vec{r}) = \dot{Q}^T \vec{r} + Q^T \dot{\vec{r}}$$

$$Q \rightarrow Q = \cos \frac{\varphi}{2} \vec{e} + \sin \frac{\varphi}{2} \vec{e} \times$$

$$Q^T \rightarrow Q^T = \cos \frac{\varphi}{2} \vec{e} - \sin \frac{\varphi}{2} \vec{e} \times = Q^*$$

$$\begin{aligned} Q^* (\dot{Q} \vec{r}) &= Q^* (\dot{Q} \vec{r} Q^* + Q \vec{r} \dot{Q}^*) Q = Q^* \dot{Q} \vec{r} + \vec{r} \dot{Q} Q^* = \\ &= (Q^* \dot{Q} \vec{r}) - (Q^* \dot{Q} \vec{r})^* = 2\vec{\omega} \end{aligned}$$

$$\frac{Q^* Q}{\vec{r}^* \vec{r}} = 1$$

1) $Q^* \dot{Q} \vec{r} = \vec{\omega} \times \vec{r}$ iščemo $\vec{\omega}$

$$Q = \cos \frac{\varphi}{2} \vec{e} + \sin \frac{\varphi}{2} \vec{e} \times$$

$$\dot{Q} = -\frac{\dot{\varphi}}{2} \sin \frac{\varphi}{2} \vec{e} + \dot{\varphi} \sin \frac{\varphi}{2} \vec{e} \times + \frac{\dot{\varphi}}{2} \vec{e} \cos \frac{\varphi}{2}$$

$$\dot{Q} \vec{r} = \left(-\frac{\dot{\varphi}}{2} \sin \frac{\varphi}{2} \vec{e} + \dot{\varphi} \sin \frac{\varphi}{2} \vec{e} \times + \frac{\dot{\varphi}}{2} \vec{e} \cos \frac{\varphi}{2} \right) \vec{r} =$$

$$= 0 - ((\vec{e} \cdot \vec{r}) \sin \frac{\varphi}{2} + (\vec{e} \cdot \vec{r}) \frac{\dot{\varphi}}{2} \cos \frac{\varphi}{2}) + \left(-\frac{\dot{\varphi}}{2} \sin \frac{\varphi}{2} \vec{r} + \sin \frac{\varphi}{2} \dot{\varphi} \vec{e} \times \vec{r} + \frac{\dot{\varphi}}{2} \cos \frac{\varphi}{2} \vec{e} \times \vec{r} \right)$$

$$\begin{aligned} Q^* (\dot{Q} \vec{r}) &= \left(\cos \frac{\varphi}{2} \vec{e} - \sin \frac{\varphi}{2} \vec{e} \times \right) \left(-\frac{\dot{\varphi}}{2} \sin \frac{\varphi}{2} \vec{e} + \dot{\varphi} \sin \frac{\varphi}{2} \vec{e} \times + \frac{\dot{\varphi}}{2} \vec{e} \cos \frac{\varphi}{2} \right) \vec{r} = \\ &= \left(\text{skalarni del} \right) + \left(-\frac{\dot{\varphi}}{2} \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \vec{r} + \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \dot{\varphi} \vec{e} \times \vec{r} + \right. \\ &\quad \left. + \frac{\dot{\varphi}}{2} \cos^2 \frac{\varphi}{2} \vec{e} \times \vec{r} + (\vec{e} \cdot \vec{r}) \sin^2 \frac{\varphi}{2} \vec{e} + (\vec{e} \cdot \vec{r}) \frac{\dot{\varphi}}{2} \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \vec{e} + \right. \\ &\quad \left. + \frac{\dot{\varphi}}{2} \sin^2 \frac{\varphi}{2} \vec{e} \times \vec{r} - \sin^2 \frac{\varphi}{2} \vec{e} \times (\vec{e} \times \vec{r}) - \frac{\dot{\varphi}}{2} \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \vec{e} \times (\vec{e} \times \vec{r}) \right) \end{aligned}$$

$$\begin{aligned} \vec{\omega} &= \vec{r} \left(-\frac{\dot{\varphi}}{2} \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} + \frac{\dot{\varphi}}{2} \cos^2 \frac{\varphi}{2} \right) + \vec{e} \left((\vec{e} \cdot \vec{r}) \sin^2 \frac{\varphi}{2} + (\vec{e} \cdot \vec{r}) \frac{\dot{\varphi}}{2} \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} - \right. \\ &\quad \left. - \frac{\dot{\varphi}}{2} \vec{e} \cdot \vec{r} \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \right) + \frac{\dot{\varphi}}{2} \vec{e} \times \vec{r} + \frac{1}{2} \sin \varphi \vec{e} \times \vec{r} \\ &\quad \quad \quad (\vec{e} \times \vec{e}) \times \vec{r} \end{aligned}$$

$$\vec{\omega} = \sin^2 \frac{\varphi}{2} ((\vec{e} \cdot \vec{r}) \vec{e} - (\vec{e} \cdot \vec{r}) \vec{e}) + \frac{\dot{\varphi}}{2} \vec{e} \times \vec{r} + \frac{1}{2} \sin \varphi \vec{e} \times \vec{r} \quad | \vec{r} \text{ izpost.}$$

$$\vec{\omega} = \left(\frac{\dot{\varphi}}{2} \vec{e} + \frac{1}{2} \sin \varphi \vec{e} + \sin^2 \frac{\varphi}{2} \vec{e} \times \vec{e} \right) \vec{r}$$

$$2\vec{\omega} = \vec{\omega} \times \vec{r} \Rightarrow \vec{\omega} = (\dot{\varphi} \vec{e} + \sin \varphi \vec{e} + (1 - \cos \varphi) \vec{e} \times \vec{e}) \checkmark = \text{ROT. VEKT}$$

2) $\vec{\omega}' = Q \vec{\omega}$

$\vec{\omega}$ rotacijski vektor za $Q \Rightarrow -\vec{\omega}$ vektor kotne hitrosti za Q^T

$$\vec{\omega}' = -(\text{rotacijski vektor za } Q^T) = -(\text{rot. vektor za } R(\vec{e}, -\varphi)) \quad (\text{vemo od zadnjice})$$

$$= -(-\dot{\varphi} \vec{e} - \sin \varphi \vec{e} - (1 - \cos \varphi) \vec{e} \times \vec{e})$$

$$\vec{\omega}' = \dot{\varphi} \vec{e} + \sin \varphi \vec{e} + (1 - \cos \varphi) \vec{e} \times \vec{e} \quad \checkmark = \text{VEKT KOTNE HITR}$$

Vektor kotne hitrosti sestavljene rotacije

$$Q = Q_2 \cdot Q_1$$

$$Q^T Q^0 \vec{r} = \vec{\omega} \times \vec{r}$$

$$\begin{aligned} (Q_2 Q_1)^T (Q_2 Q_1)^0 \vec{r} &= (Q_1^T Q_2^T) (Q_2^0 Q_1^0 + Q_2 Q_1^0) \vec{r} = \\ &= \underbrace{(Q_1^T Q_2^T Q_2^0 Q_1^0) \vec{r}}_{Q_1^T (\vec{\omega}_2 \times Q_1 \vec{r})} + \underbrace{(Q_1^T Q_2^T Q_2 Q_1^0) \vec{r}}_{Q_1^T \cdot Q_1^0 \vec{r}} = \\ &= \underbrace{Q_1^T \vec{\omega}_2 \times \vec{r}}_{Q_1^T \vec{\omega}_2 \times \vec{r}} + \underbrace{\vec{\omega}_1 \times \vec{r}}_{\vec{\omega}_1 \times \vec{r}} \end{aligned}$$

Trditev Rotacijski vektor ust. rotacije $Q = Q_2 Q_1$

$$\vec{\omega} = \vec{\omega}_1 + Q_1^T \vec{\omega}_2$$

Vektor kotne hitr. pa je $\vec{\omega}' = Q_2 \vec{\omega}_1' + \vec{\omega}_2'$

Dokaz $\vec{\omega}' = (Q_2 Q_1) \vec{\omega} = Q_2 (Q_1 \vec{\omega}) = Q_2 (Q_1 \vec{\omega}_1 + Q_1^T \vec{\omega}_2) =$
 $= \vec{\omega}_2' + Q_2 \vec{\omega}_1'$

$Q \rightarrow \vec{\omega}'$ ni homeom. med grupama

$$Q_2 Q_1 \mapsto \vec{\omega}_1' + \vec{\omega}_2'$$

Q notacija

$$P_1' = P_0'(t) + Q(t)(P_1 - P_0)$$

$$P_2' = P_0'(t) + Q(t)(P_2 - P_0)$$

$$\frac{P_2' - P_1'}{\vec{\alpha}'} = \frac{Q(t)(P_2 - P_1)}{\vec{\alpha}}$$

$$\vec{\alpha} = P_2 - P_1$$

$$\vec{\alpha}' = P_2' - P_1'$$

$$\begin{aligned} \vec{v}' &= \frac{dP'}{dt} = \frac{d}{dt} (P_0' + Q(P - P_0)) = \\ &= \underbrace{\frac{dP_0'}{dt}}_{\vec{v}_0'} + \underbrace{\dot{Q}(P - P_0)}_{Q(Q^T \dot{Q})(P - P_0)} + \underbrace{Q \frac{dP}{dt}}_{Q \vec{v}_{rel}} \\ &= \vec{v}_0' + Q(\vec{\omega} \times (P - P_0)) + Q \vec{v}_{rel} \\ &= \vec{v}_0' + \vec{\omega}' \times Q(P - P_0) + Q \vec{v}_{rel} \end{aligned}$$

$$\vec{v}' = \vec{v}_0' + \vec{\omega}' \times Q(P - P_0) + Q \vec{v}_{rel} = \vec{v}_0' + \vec{\omega}' \times \vec{\xi}' + \vec{v}_{rel}' = \vec{v}'$$

$$\vec{\xi} = P - P_0$$

$$\vec{\xi}' = Q \vec{\xi}$$

translacijska
hitrost

$\vec{v}_0' + (\vec{\omega}' \times \vec{\xi}')$... sistemska
hitrost

REL. KS	ABS. KS	medwise od koord. sistema
P	P'	"
$\vec{a}$	$\vec{a}' = Q \vec{a}$	potem je vektor koord. medwise abs
$P = P(t)$	$P' = P'(t)$	
$\vec{v}_{rel} = \dot{P}$	$\vec{v}' = \dot{P}'$	vektor hitr. v AKS.
	$\vec{v}'_{rel} = Q \vec{v}_{rel}$	vektor hitr. v AKS.

$$\vec{v}_0 = Q^T \vec{v}'$$

$$\begin{aligned} \vec{\xi} &= P - P_0 \\ \dot{\vec{\xi}} &= \dot{P} \end{aligned}$$

$$\vec{v}' = Q (\vec{v}_0 + \vec{\omega} \times \vec{\xi} + \vec{v}_{rel})$$

$$\vec{v} = \vec{v}_0 + \vec{\omega} \times \vec{\xi} + \vec{v}_{rel}$$

vektor abs. hitrosti zapisan v RKS.

$$\bullet \vec{a}' = \frac{d\vec{v}'}{dt} = \frac{d}{dt} (Q (\vec{v}_0 + \vec{\omega} \times \vec{\xi} + \vec{v}_{rel})) =$$

$$\frac{d}{dt} (\vec{v}' + Q (\vec{\omega} \times \vec{\xi} + \vec{v}_{rel})) = \frac{d\vec{v}_0'}{dt} + \dot{Q} (\vec{\omega} \times \vec{\xi} + \vec{v}_{rel}) + Q (\dot{\vec{\omega}} \times \vec{\xi} + \vec{\omega} \times \dot{\vec{\xi}} + \dot{\vec{v}}_{rel}) =$$

$$= \vec{a}_0' + \dot{Q} (\vec{\omega} \times (\vec{\omega} \times \vec{\xi} + \vec{v}_{rel})) + Q (\dot{\vec{\omega}} \times \vec{\xi} + \vec{\omega} \times \dot{\vec{v}}_{rel} + \vec{a}_{rel})$$

$$\vec{a}_0 = Q^T \vec{a}_0'$$

$$\vec{a}' = Q (\vec{a}_0 + \vec{\omega} \times (\vec{\omega} \times \vec{\xi}) + \dot{\vec{\omega}} \times \vec{\xi} + 2 \vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel})$$

$$\vec{a}' = \vec{a}_0' + (\vec{\omega}' \times (\vec{\omega}' \times \vec{\xi}')) + (\dot{\vec{\omega}})' \times \vec{\xi}' + 2 \vec{\omega}' \times \vec{v}_{rel}' + \vec{a}_{rel}'$$

Centrifugalni pospešek

Coriolisov pospešek

relativni pospešek

pospešek abs. gibanja, zapisan v abs. koord. sist.

pospešek abs. gibanja v relat. koord. sist.

$$\vec{a} = \vec{a}_0 + \vec{\omega} \times (\vec{\omega} \times \vec{\xi}) + \dot{\vec{\omega}} \times \vec{\xi} + 2 \vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel}$$

$$\vec{a}_s' = \vec{a}_0' + \vec{\omega}' \times (\vec{\omega}' \times \vec{\xi}') + \dot{\vec{\omega}}' \times \vec{\xi}'$$

... sistemski pospešek

$$\vec{a}_c = 2 \vec{\omega}' \times \vec{v}_{rel}'$$

... Coriolisov pospešek

$$\begin{aligned} (\dot{\vec{\omega}})' &= Q \dot{\vec{\omega}} \\ (\ddot{\vec{\omega}})' &= \ddot{Q} \vec{\omega} \end{aligned}$$

obkret

$$(\vec{\omega})' = \frac{d\vec{\omega}'}{dt} = \frac{d}{dt} Q \vec{\omega} = \dot{Q} \vec{\omega} + Q \dot{\vec{\omega}} =$$

$$= Q \dot{Q}^T Q \vec{\omega} + Q \dot{\vec{\omega}} = Q (\vec{\omega} \times \vec{\omega}) + Q \dot{\vec{\omega}} = Q \dot{\vec{\omega}}$$

Trditelj:

$$(\vec{\omega}')' = (\ddot{\vec{\omega}})' = \ddot{\vec{\omega}}'$$

$$\bullet m \vec{a}' = \vec{F}' \text{ Newtonova enačba v abs. KS}$$

$$\bullet m \vec{a}_{rel}' = \vec{F}_{rel}' \text{ Newtonova enačba v rel. KS}$$

$$\vec{F}_{rel}' = m \vec{a}_{rel}' = m (\vec{a} - \vec{a}_0 - \vec{\omega} \times (\vec{\omega} \times \vec{\xi}) - \dot{\vec{\omega}} \times \vec{\xi} - 2 \vec{\omega} \times \vec{v}_{rel}) =$$

$$= \vec{F}' - m \vec{a}_0 - m \vec{\omega} \times (\vec{\omega} \times \vec{\xi}) - m \dot{\vec{\omega}} \times \vec{\xi} - 2 m \vec{\omega} \times \vec{v}_{rel}$$

$$\bullet -2 m \vec{\omega} \times \vec{v}_{rel} = \vec{F}_c \text{ ... Coriolisova sila}$$

$$\bullet -m \dot{\vec{\omega}} \times \vec{\xi} \text{ ... inercialna sila rotacije}$$

$$\bullet -m (\vec{\omega} \times (\vec{\omega} \times \vec{\xi})) \text{ ... Centrifugalna sila}$$

$$\bullet -m \vec{a}_0 \text{ ... sila translacije}$$

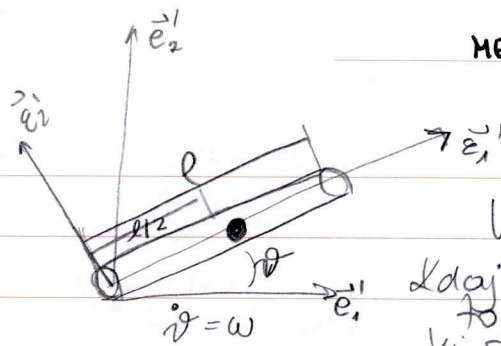
Ten silam opazovalec v RKS pravi fiktivne.

$$\begin{aligned} \vec{a} &= Q^T \vec{a}' \quad m \vec{a} = \vec{F} \\ \vec{F} &= Q^T \vec{F}' \end{aligned}$$

Newton. e. v abs. gibanju v rel. KS.

PRIMER 1

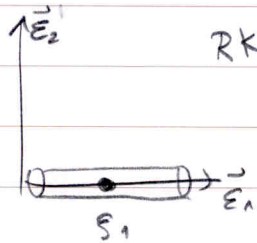
votla cev

slika v
abs. ~~rel.~~ koord sist

$\vec{F}' = ?$ kakšna sila deluje na točko v AKS?

V času $t=0$ vez popusti

Kdaj in s kakšno hitrostjo točka zapusti cev, ki se vrtilo bo znotraj? Kakšna hitrostjo?



RKS

$$\vec{\xi} = \xi_1 \vec{e}_1$$

$$\vec{v}_{rel} = \dot{\vec{\xi}} = \dot{\xi}_1 \vec{e}_1$$

$$\vec{a}_{rel} = \ddot{\vec{\xi}} = \ddot{\xi}_1 \vec{e}_1$$

$$m \vec{a}_{rel} = \vec{F} - m \vec{a}_0 - m \vec{\omega} \times (\vec{\omega} \times \vec{\xi}) - m \vec{\omega} \times \dot{\vec{\xi}} - 2 m \vec{\omega} \times \vec{v}_{rel}$$

$$\vec{F} = Q \vec{F}'$$

$$\vec{F}' = \vec{F} \cdot \vec{e}_2' ; \text{ ker je gibanje brez trenja}$$

$$\text{Če je trenje, je } \vec{F}' = F_1 \vec{e}_1 + F_2 \vec{e}_2$$

sila pralica

$$\vec{F} \cdot \vec{e}_2'$$

$$\left. \begin{aligned} \vec{e}_1' &= Q \vec{e}_1 \\ \vec{F}' &= Q \vec{F} \end{aligned} \right\} \vec{F} = F_1 \vec{e}_1 + F_2 \vec{e}_2$$

- $\vec{a}_0 =$ pospešek izhodišča RKS zapisan v AKS.

Izhodišči se ne premikajo $\Rightarrow \vec{a}_0 = 0$. (translacija ni!)

- $\vec{\omega} = \vec{\omega}$, ker se pralica vrtila enakomerno ($\vec{\omega} = \omega_0 \vec{e}_3 = \omega_0 \vec{e}_3' = \omega_0 \vec{e}_3'' = \omega_0 \vec{e}_3'''$)

kotni pospešek

$$\vec{\omega} \times \vec{\xi} = \omega_0 \vec{e}_3 \times \xi_1 \vec{e}_1 = \omega_0 \xi_1 \vec{e}_2$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{\xi}) = \omega_0 \vec{e}_3 \times (\omega_0 \xi_1 \vec{e}_2) = -\omega_0^2 \xi_1 \vec{e}_1$$

$$\vec{\omega} \times \vec{v}_{rel} = \omega_0 \vec{e}_3 \times \dot{\xi}_1 \vec{e}_1 = \omega_0 \dot{\xi}_1 \vec{e}_2$$

$$\Rightarrow m \ddot{\xi}_1 \vec{e}_1 = F_1 \vec{e}_1 + F_2 \vec{e}_2 + m \omega_0^2 \xi_1 \vec{e}_1 - 2 m \omega_0 \dot{\xi}_1 \vec{e}_2$$

$$m \ddot{\xi}_1 = F_1 + m \omega_0^2 \xi_1 \quad \text{2 enačbi (po } \vec{e}_1 \text{ in } \vec{e}_2)$$

$$0 = F_2 - 2 m \omega_0 \dot{\xi}_1$$

$$\text{Če je trenje } F_1 = -k F_2 ; k > 0$$

$$m \ddot{\xi}_1 = -2 k m \omega_0 \dot{\xi}_1 + m \omega_0^2 \xi_1$$

Če ni trenja:

$$m \ddot{\xi}_1 = m \omega_0^2 \xi_1$$

$$\text{DN: } \vec{\omega} = \frac{d\vec{\omega}}{dt} = \text{konstanten}$$

$$\ddot{\xi}_1 - \omega_0^2 \xi_1 = 0 \quad (\text{če } \neq 0 \text{ enačba harmoničn. nihanja})$$

dij. en. 2. reda

rešitev

$$\xi_1 = A \cosh(\omega_0 t) + B \sinh(\omega_0 t)$$

$$\dot{\xi}_1 = A \omega_0 \sinh(\omega_0 t) + B \omega_0 \cosh(\omega_0 t)$$

$$\ddot{\xi}_1 = \omega_0^2 \xi_1$$

A, B iz zad. pogojev: $\dot{\xi}_1(t=0) = 0 \Rightarrow B = 0$
 $\xi_1(t=0) = \frac{l}{2} \Rightarrow A = \frac{l}{2}$
 $\Rightarrow \underline{\xi_1 = \frac{l}{2} \operatorname{ch}(\omega_0 t)}$

$$l = \frac{l}{2} \operatorname{ch}(\omega_0 t_1)$$

$$2 = \operatorname{ch} \omega_0 t_1$$

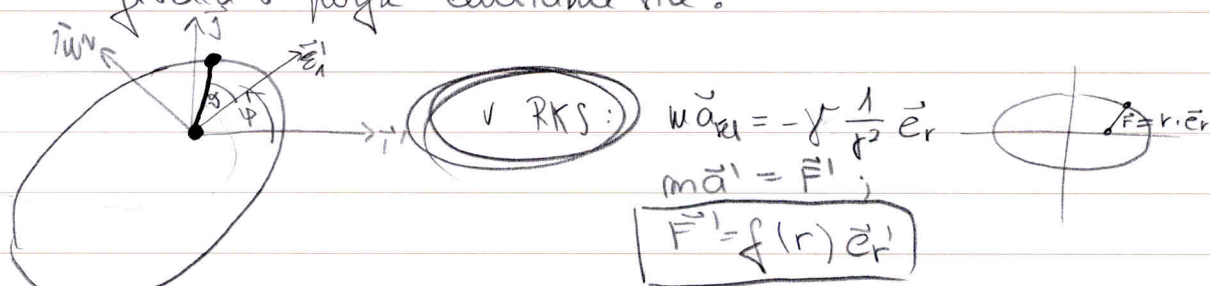
$$t_1 = \frac{1}{\omega_0} \operatorname{Arch} 2$$

$$v_1 = \frac{l}{2} \omega_0 \operatorname{sh} \omega_0 t_1 = \frac{l}{2} \omega_0 \sqrt{\operatorname{ch}^2 \omega_0 t_1 - 1}$$

$$\operatorname{ch}^2 z - \operatorname{sh}^2 z = 1$$

$$\underline{v_1 = \frac{l}{2} \omega_0 \sqrt{3}}$$

PRIMER 2 V RKS se točka giblje po elipsi s konst. pl. hitrostjo. Določiti potacijo RKS, tako da se točka v AKS gibala v pogoj centrirane sile.



$$\vec{F}_{rel} = \vec{F} - (m \ddot{\mathbf{a}}_0) - m \ddot{\omega} \times \vec{\xi} - m \ddot{\omega} \times (\vec{\omega} \times \vec{\xi}) - 2m \ddot{\omega} \times \vec{v}_{rel}$$

$$\vec{F} = \vec{F}_{rel} + m \ddot{\omega} \times \vec{\xi} + m \ddot{\omega} \times (\vec{\omega} \times \vec{\xi}) + 2m \ddot{\omega} \times \vec{v}_{rel}$$

$$\ddot{\omega} = \omega \cdot \vec{k}' = \omega \cdot \vec{e}_3$$

$$\vec{\xi} = \xi_1 \vec{e}_1 + \xi_2 \vec{e}_2$$

$$\ddot{\omega} \times \vec{\xi} = \ddot{\omega} \vec{e}_3 \times (\xi_1 \vec{e}_1 + \xi_2 \vec{e}_2) = \xi_1 \ddot{\omega} \vec{e}_2 - \xi_2 \ddot{\omega} \vec{e}_1$$

$$\ddot{\omega} \cdot \vec{\xi} = \ddot{\omega} \vec{e}_3 \cdot r \vec{e}_r = \ddot{\omega} r \vec{e}_\varphi$$

$$\ddot{\omega} \times \vec{\xi} = \ddot{\omega} r \vec{e}_\varphi$$

$$\ddot{\omega} \times (\ddot{\omega} \times \vec{\xi}) = \ddot{\omega}^2 r \vec{e}_3 \times \vec{e}_\varphi = -\ddot{\omega}^2 r \vec{e}_r$$

$$\ddot{\omega} \times \vec{v}_{rel} = \ddot{\omega} \cdot \vec{e}_3 \times (r \dot{\vec{e}}_r + r \dot{\varphi} \vec{e}_\varphi) = \ddot{\omega} r \dot{\vec{e}}_\varphi - \ddot{\omega} r \dot{\varphi} \vec{e}_r$$

$$\vec{F} = -\frac{\gamma}{r^2} \vec{e}_r + m \ddot{\omega} r \vec{e}_\varphi - m \ddot{\omega}^2 r \vec{e}_r + 2m \ddot{\omega} (r \dot{\vec{e}}_\varphi - r \dot{\varphi} \vec{e}_r)$$

$$F = \vec{e}_r \left(-\frac{\gamma}{r^2} - m \ddot{\omega}^2 r - 2m \ddot{\omega} r \dot{\varphi} \right) + \vec{e}_\varphi (m \ddot{\omega} r + 2m \ddot{\omega} r \dot{\varphi})$$

$$\vec{F} =$$

$$m \ddot{\omega} r + 2m \ddot{\omega} r \dot{\varphi} = 0$$

$$\left(\frac{\ddot{\omega}}{\omega} + 2 \frac{\dot{\varphi}}{r} \right) = 0 \Rightarrow \log \omega + 2 \log r = \log c$$

$$\omega r^2 = c \Rightarrow \omega = \frac{c}{r^2}$$

da bi bila to
v rad. smeri
 $\vec{F}' = f(r) \vec{e}_r$
 $\vec{F} = f(r) \vec{e}_r$

→ RKS se me vrti enakomerno.

$$\vec{F} = \vec{e}_r \left(-\frac{\gamma}{r^2} - m \frac{\mu^2}{r^3} - 2m \left(\frac{\mu}{r^2} \right) r \left(\frac{C_0}{r^2} \right) \right)$$

$$\vec{F} = \underbrace{-\frac{\gamma}{r^2} \vec{e}_r}_{\text{gravitacijska sila}} + \underbrace{\left(-\frac{1}{r^3} (m\mu^2 + 2m\mu C_0) \right) \vec{e}_r}_{\text{perturbacija}}$$

$$r^2 \dot{\vartheta} = C_0$$

VRKS:

$$r = \frac{p}{1 + \varepsilon \cos \vartheta}$$

AKS:

(r, ψ)

polarne koord. v AKS ⇒

$$r = \frac{p}{1 + \varepsilon \cos \left(\frac{1}{1 + \frac{\mu}{C_0}} \right) \psi}$$

$$\psi = \vartheta + \varphi$$

$$\dot{\psi} = \dot{\vartheta} + \dot{\varphi} = \dot{\vartheta} + \omega = \dot{\vartheta} + \frac{\mu}{r^2} \quad ; \quad \dot{\vartheta} = \frac{C_0}{r^2}$$

$$= (C_0 + \mu) \frac{1}{r^2} \quad \left(C_0 = \text{dugina pol. ošč. hitrost v RKS} \right)$$

v AKS velja:

$$\psi = \frac{C_0'}{r^2}$$

$$C_0' = C_0 + \mu$$

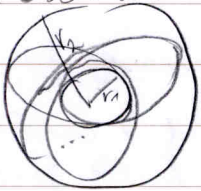
$$\dot{\psi} = \frac{C_0'}{r^2} = \frac{C_0' \cdot C_0}{C_0 r^2} = \frac{C_0 + \mu}{C_0} \cdot \dot{\vartheta}$$

primerno:

$$r = \frac{p}{1 + \varepsilon \cos \vartheta}$$

$$\psi = \frac{1 + \frac{\mu}{C_0}}{1} \vartheta + \text{konst} \Rightarrow \psi = \left(1 + \frac{\mu}{C_0} \right) \vartheta$$

$$\vartheta = \frac{1}{1 + \frac{\mu}{C_0}} \psi$$



$$r_1 = a(1 - \varepsilon)$$

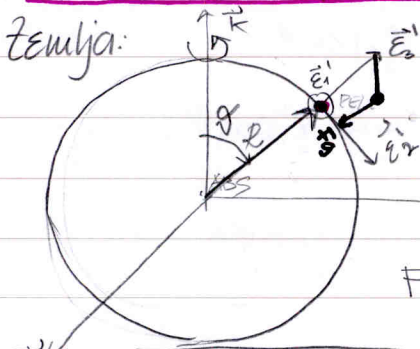
$$r_2 = a(1 + \varepsilon)$$

PRIMER 3

Faucaultovo nihalo

1852

zemlja:

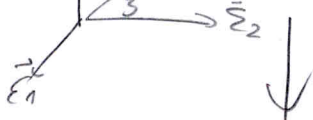


zhod. mesta enak

$$\vec{a}' = \vec{a}_0' + \vec{\omega}' \times \vec{r}' \times \vec{\omega}' \times (\vec{\omega}' \times \vec{r}') + 2\vec{\omega}' \times \vec{v}' + \vec{a}_{rel}$$

$$\vec{a}_0' = \vec{\omega} \times (\vec{\omega} \times \vec{R}) \quad (\text{memorizirani za } \vec{R})$$

$$(0,0,0) \quad \vec{P}(\xi_1, \xi_2, \xi_3) \quad |\vec{P} - \vec{P}_0| = l$$



Alkita

$$\vec{\omega}' = \vec{\omega} = \omega \vec{e}' \quad (\text{rotacija okoli stalne osi})$$

$$\omega = \frac{2\pi}{24 \cdot 3600 \text{ s}} = 7.10^{-5} \frac{1}{\text{s}} < 10^{-4} \frac{1}{\text{s}}$$

Newt. enačba

$$m \vec{a}_{rel} = \vec{F} - m \vec{a}_0' - \underbrace{m(\vec{\omega}) \times \vec{r}}_0 - \underbrace{m \vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{velikost } 10^{-8} \text{ (zanemarljivo)}} - 2m \vec{\omega} \times \vec{v}_{rel}$$

$$\vec{F} = \vec{F}_g + \vec{T} \quad \uparrow \text{ sila vrvice}$$

$$m\ddot{\vec{r}} = \vec{F}_g + \vec{T} - m\vec{\omega} \times (\vec{\omega} \times \vec{R}) - 2m\vec{\omega} \times \vec{v}_{rel}$$

↑ sila gravitacije

$$m\ddot{\vec{r}} = -mg\vec{e}_3 + \vec{T} - 2m\vec{\omega} \times \vec{v}_{rel} \quad \boxed{\vec{F}_g - m\vec{\omega} \times (\vec{\omega} \times \vec{R}) = m\vec{g}} = -mg\vec{e}_3'$$

$$\vec{T} = T \frac{1}{l} (\vec{P} - \vec{P}_0) = \frac{T}{l} (-\xi_1 \vec{e}_1 - \xi_2 \vec{e}_2 + (l - \xi_3) \vec{e}_3)$$

$$\vec{v}_{rel} = \dot{\xi}_1 \vec{e}_1 + \dot{\xi}_2 \vec{e}_2 + \dot{\xi}_3 \vec{e}_3$$

$$\vec{\omega}' = \vec{\omega} \vec{e}' = (\cos\vartheta \vec{e}_3' + \sin\vartheta \vec{e}_2') \omega$$

$$\vec{\omega} = \omega (\cos\vartheta \vec{e}_3 + \sin\vartheta \vec{e}_2)$$

$$\vec{\omega} \times \vec{v}_{rel} = \omega (\cos\vartheta \dot{\xi}_1 \vec{e}_3 - \cos\vartheta \dot{\xi}_2 \vec{e}_1 + \sin\vartheta \dot{\xi}_1 \vec{e}_3 + \sin\vartheta \dot{\xi}_3 \vec{e}_1)$$

$$m\ddot{\xi}_1 = -\frac{T}{l}\xi_1 - 2m\omega(-\dot{\xi}_2 \cos\vartheta + \dot{\xi}_3 \sin\vartheta)$$

$$m\ddot{\xi}_2 = -\frac{T}{l}\xi_2 - 2m\omega(\dot{\xi}_1 \cos\vartheta)$$

$$m\ddot{\xi}_3 = \frac{T}{l}(l - \xi_3) - mg + 2m\omega\dot{\xi}_1 \sin\vartheta$$

$$+ |\vec{P} - \vec{P}_0| = l$$

system 4 eqs
 $T, \xi_1, \xi_2, \xi_3 = ?$

$$l \gg 1 \text{ (low)} + \text{majhni odklaji} \Rightarrow \ddot{\xi}_3, \dot{\xi}_3 = 0$$

$$T(1 - \frac{\xi_3}{l}) = T$$

$$\Rightarrow T = mg - 2m\omega\dot{\xi}_1 \sin\vartheta$$

$$\Rightarrow m\ddot{\xi}_1 = -\frac{1}{l}(mg - 2m\omega\dot{\xi}_1 \sin\vartheta)\xi_1 + 2m\omega(+\dot{\xi}_2 \cos\vartheta)$$

$$m\ddot{\xi}_2 = -\frac{1}{l}(mg - 2m\omega\dot{\xi}_1 \sin\vartheta)\xi_2 - 2m\omega\dot{\xi}_1 \cos\vartheta$$

nelinearni sistem enačb

$$2m\omega\dot{\xi}_1 \xi_1 \sin\vartheta \quad | \xi_1 | \ll 1, | \xi_2 | \ll 1 \Rightarrow | \dot{\xi}_1 | \ll 1$$

ga zamenamo v prve enačbi
 imamo v 2. enačbi

$$\Rightarrow \ddot{\xi}_1 = -\frac{g}{l}\xi_1 + 2\omega\dot{\xi}_2 \cos\vartheta$$

$$\ddot{\xi}_2 = -\frac{g}{l}\xi_2 - 2\omega\dot{\xi}_1 \cos\vartheta$$

1. i 2. } 2 linearni enačbi

$$\text{Vpeljemo } \frac{g}{l} = k^2; \quad \omega \cos\vartheta = \beta, \quad z = \xi_1 + i\xi_2$$

$$\Rightarrow \ddot{z} = -k^2 z + 2\beta(\xi_2 - i\xi_1)$$

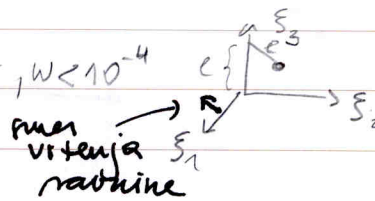
$$\boxed{\ddot{z} + 2i\beta \dot{z} + k^2 z = 0}$$

$$\ddot{z} + 2i\beta \dot{z} + k^2 z = 0, \quad k^2 = \frac{g}{l}, \quad \beta = \omega \cos \vartheta, \quad \omega < 10^{-4}$$

Rešujemo z postavdom: $z = e^{\lambda t}$

$$\dot{z} = \lambda e^{\lambda t}$$

$$\ddot{z} = \lambda^2 e^{\lambda t}$$



$$\lambda^2 e^{\lambda t} + 2i\beta \lambda e^{\lambda t} + k^2 e^{\lambda t} = 0$$

$$\lambda^2 + 2i\beta \lambda + k^2 = 0$$

$$\lambda_{1,2} = -i\beta \pm \sqrt{-\beta^2 + k^2} \Rightarrow z = A e^{\lambda_1 t} + B e^{\lambda_2 t} \quad \text{splošna rešitev}$$

$$= i(-\beta \pm \sqrt{\beta^2 + k^2}) \quad ; \quad \sqrt{\beta^2 + k^2} = k \sqrt{1 + \left(\frac{\beta}{k}\right)^2} = k \left(1 + \frac{1}{2} \left(\frac{\beta}{k}\right)^2 + O\left(\left(\frac{\beta}{k}\right)^4\right)\right)$$

$$\Rightarrow \lambda_{1,2} = i \left(-\beta \pm \left(k + \frac{1}{2} \frac{\beta^2}{k}\right)\right) = i(-\beta \pm k)$$

↓ zamenam.

$$z = A e^{i(-\beta - k)t} + B e^{i(-\beta + k)t}$$

Začetni pogoji: $\xi_1(t=0) = A_0$

$\xi_2(t=0) = 0$

$z(t=0) = A_0 \in \mathbb{R}$

t=0: $A_0 = A + B$

$\dot{\xi}_1(t=0) = \dot{\xi}_2(t=0) = 0$

odv.: $0 = i(-\beta - k)A + i(-\beta + k)B$

$$B = \frac{\beta + k}{k - \beta} A = \frac{1 + \left(\frac{\beta}{k}\right)^0}{1 - \left(\frac{\beta}{k}\right)^0} A = A \Rightarrow A = B = \frac{A_0}{2}$$

$$z = \frac{A_0}{2} \left(e^{i(-\beta - k)t} + e^{i(-\beta + k)t} \right) =$$

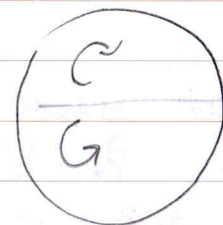
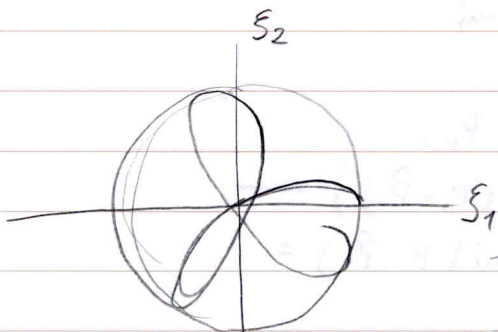
$$= \frac{A_0}{2} e^{-i\beta t} (e^{-ikt} + e^{ikt}) = A_0 e^{-i\beta t} \cos kt \quad ; \quad |z| = |A_0| |\cos kt|$$

$$\xi_1 = A_0 \cos \beta t \cos kt = A_0 \cos(\omega \cos \vartheta) t \cdot \sin kt$$

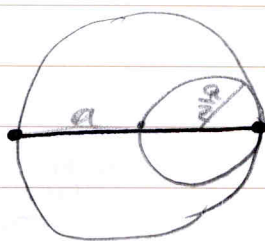
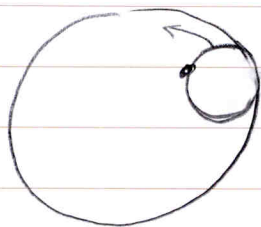
$$\xi_2 = -A_0 \sin \beta t \cos kt = -A_0 \sin(\omega \cos \vartheta) t \cos kt$$

→ pravilna puščanja se vrsti s kotno hitrostjo $\omega \cos \vartheta$.

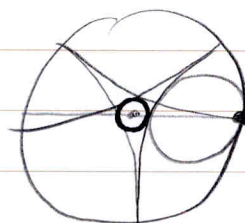
Na severni polobi v smeri urnega kazalca, na ekvatorju se ne vrsti, na južni pa v naspr. smeri, na polu s kotno hitr. 2-krat.



Bez aproksimacije $\left(\frac{\beta}{k} = 0\right)$, dobimo hipocikloido



gibanje
po
X-OX



$\ddot{\xi}_1 = 0, \ddot{\xi}_2 = 0$
 $\frac{\ddot{\xi}_2}{2} = 0$ } imamo max. gibanje po pravm. ξ_1, ξ_2

$$|\xi_1 \cdot \xi_2| \ll 1, \quad |\dot{\xi}_1 \cdot \dot{\xi}_2| \ll 1$$

$$\dot{\xi}_1 = -A_0 \rho \sin \rho t \sin kt + A_0 \cos \rho t (-k \sin kt)$$

$$1 \gg \xi_1 \cdot \dot{\xi}_1 = (A_0^2) \left(-\rho \frac{1}{2} \sin 2\rho t \cdot \cos^2 kt - k \cos^2 \rho t \frac{1}{2} \sin 2kt \right)$$

možno bita A_0 majhen (=odmikanje)

SISTEM MATERIALNIH TOČEK

$P_i, m_i; i=1, \dots, N$

$$m_i \ddot{P}_i = \vec{F}_i + \sum_{j=1}^N \vec{F}_{ji}$$

$\vec{F}_i$ zunanja sila na P_i

$\vec{F}_{ji}$ sila P_j na P_i (sila j -te točke na i -to)

Imamo 3N enačb.

$$\vec{F}_{ji} = k \left(|P_i - P_j| - c \right) \frac{P_i - P_j}{|P_i - P_j|}$$

$\leftarrow$ neutralna razd.

Masno središče (=težišče)

DEF. $P_* = P_0 + \frac{1}{m} \sum_{i=1}^N m_i (P_i - P_0) \quad ; \quad m = \sum_{i=1}^N m_i$

TRDITEV Masno središče je neodvisno od P_0 .

DOKAZ $\hat{P}_0 + \frac{1}{m} \sum m_i (P_i - \hat{P}_0) = ((P_i - P_0) + (P_0 - \hat{P}_0))$ ~~celo~~

$$= \hat{P}_0 + \frac{1}{m} \sum m_i (P_i - P_0) + \frac{1}{m} \sum_{i=1}^N m_i (P_0 - \hat{P}_0) =$$

$$= P_0 + \frac{1}{m} \sum m_i (P_i - P_0) = P_*$$

$$m_i \ddot{\vec{P}}_i = \vec{F}_i + \sum_j \vec{F}_{ji} \quad | \quad \sum_i$$

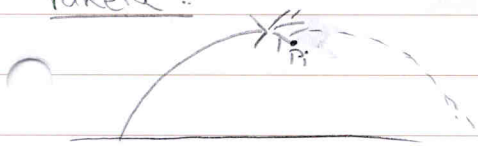
$$\underbrace{\sum_i m_i \ddot{\vec{P}}_i}_{m \ddot{\vec{P}}_*} = \underbrace{\sum_i \vec{F}_i}_{\substack{\vec{F} \text{ rezultanta} \\ \text{vseh zun.} \\ \text{sile}}} + \underbrace{\sum_{i,j} \vec{F}_{ji}}_{\substack{\text{rezult.} \\ \text{notr. sil} \\ \parallel \\ 0 \leftarrow}}$$

$$\left. \begin{array}{l} \dot{\vec{P}}_* = \frac{1}{m} \sum m_i \dot{\vec{P}}_i \quad (\text{hitrost m.s.}) \\ \ddot{\vec{P}}_* = 0 \end{array} \right\} \left. \begin{array}{l} \ddot{\vec{P}}_* = \frac{1}{m} \sum m_i \ddot{\vec{P}}_i \quad (\text{posp. m.s.}) \\ \end{array} \right|$$

3. Newton zakon
($\vec{F}_{ij} = -\vec{F}_{ji}$)

$$\Rightarrow \boxed{m \ddot{\vec{P}}_* = \vec{F}} \quad \text{enaka gibanja m.s.}$$

raketa



Vrtna količina

$$\vec{\ell}_i(0) = \vec{\ell}_i(P_i; 0) = (\vec{P}_i - 0) \times m_i \dot{\vec{P}}_i$$

vrtna kol. sistema

$$\vec{L} = \sum \vec{\ell}_i(0)$$

$$\frac{d\vec{L}}{dt} = \sum \dot{\vec{\ell}}_i(0) = \sum (\dot{\vec{P}}_i \times m_i \dot{\vec{P}}_i + (\vec{P}_i - 0) \times m_i \ddot{\vec{P}}_i) = \sum (\vec{P}_i - 0) \times (\vec{F}_i + \sum_j \vec{F}_{ji}) =$$

$$= \underbrace{\sum_i (\vec{P}_i - 0) \times \vec{F}_i}_{\vec{N}(0)} + \underbrace{\sum_{i,j} (\vec{P}_i - 0) \times \vec{F}_{ji}}_{\vec{N}_n(0)}$$

moment sil $\vec{F}_i$ na P_i

$\vec{N}(0)$... rezultanta pravok. zun. sil
 $\vec{N}_n(0)$... moment notr. sil

$$\Rightarrow \boxed{\dot{\vec{L}}(0) = \vec{N}(0) + \vec{N}_n(0)}$$

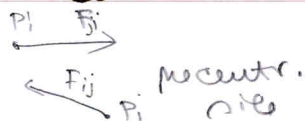
Kdaj $\vec{N}_n(0) = 0$?

DEF $\vec{F}_{ji} = \vec{F}_{ji}(P_i, P_j)$ je centralna sila, če je

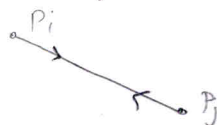
$$\vec{F}_{ji} = F_{ji} \frac{\vec{P}_i - \vec{P}_j}{|\vec{P}_i - \vec{P}_j|}$$

skalarni faktor

Centr. sila deluje v smeri zveznice med P_i in P_j



Akta



TRDITEV

DOKAZ

Če so notranje sile centralne je mawr notr sil = 0.

$$\sum_{ij} (P_i - O) \times \vec{F}_{ji} = \frac{1}{2} \left(\sum_{ij} (P_i - O) \times \vec{F}_{ji} + \sum_{ji} (P_j - O) \times \vec{F}_{ij} \right) =$$

$$= \frac{1}{2} \left(\sum_{ij} ((P_i - O) - (P_j - O)) \times \vec{F}_{ji} \right) = \frac{1}{2} \sum_{ij} (P_i - P_j) \times \vec{F}_{ji}$$

večt = 0 mod. vzpor. vekt.
ker $\vec{F}_{ji}$ centralno

IZREK (o vrtilni količini)

Če so notranje sile centralne, je odhod vrtil. kol. enak rezultanti mawrov zun. sil.

$$\sum_{i=1}^N (P_i - O) \times \vec{F}_i = \frac{d\vec{L}}{dt}(O) = \vec{N}(O)$$

$$m \ddot{P}_* = \vec{F}$$

0 fiksna točka

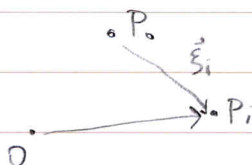
3+3 = 6 enačb

3N parametrov

$$\vec{L}(O) = \sum_{i=1}^N (P_i - O) \times m_i \dot{P}_i$$

$$\vec{L}(P_0) = \sum_{i=1}^N (P_i - P_0) \times m_i \dot{P}_i = \vec{L}(O) + (O - P_0) \times \sum_{i=1}^N m_i \dot{P}_i$$

→ vrtilna kol. preko poljubne točke: $\vec{L}(P_0) = \vec{L}(O) + (O - P_0) \times m \dot{P}_*$



$$\vec{\xi}_i = P_i - P_0$$

$$\dot{P}_i = \dot{P}_0 + \dot{\xi}_i$$

$$(P_i - O) = (P_0 - O) + \xi_i$$

$$\vec{L}(P_0) = \sum_{i=1}^N \xi_i \times m_i \dot{\xi}_i$$

vrtilna količina
relativnega gibanja
glede na P_0

$$\vec{L}(P_0) = \sum \xi_i \times m_i \dot{\xi}_i = \sum (P_i - P_0) \times m_i (\dot{P}_i - \dot{P}_0) =$$

$$= \sum \dots + \sum \dots = \vec{L}(P_0) - \sum_{i=1}^N (P_i - P_0) \times m_i \dot{P}_0$$

$$\rightarrow \sum (P_i - P_0) m_i = m(P_* - P_0)$$

$$\vec{L}(P_0) = \vec{L}(P_0) - m(P_* - P_0) \times \dot{P}_0$$

$$\frac{d\vec{L}(O)}{dt} = \vec{N}(O)$$

$$\frac{d}{dt}(\vec{L}(P_0) - (O-P_0) \times m \dot{\vec{P}}_*) = \frac{d}{dt}(\vec{L}(P_0) + m(P_*-P_0) \times \dot{\vec{P}}_0 - (O-P_0) \times m \dot{\vec{P}}_*)$$

$$= \frac{d\vec{L}(P_0)}{dt} + m \cancel{\dot{\vec{P}}_*} \times \dot{\vec{P}}_0 - (-\dot{\vec{P}}_0) \times m \dot{\vec{P}}_* = (O-P_0) \times m \ddot{\vec{P}}_* + m(P_*-P_0) \times \ddot{\vec{P}}_0 =$$

$$\frac{d\vec{L}(O)}{dt} = \frac{d\vec{L}(P_0)}{dt} + (P_*-P_0) \times m \ddot{\vec{P}}_* - (O-P_0) \times \vec{F} \quad \leftarrow \text{rezult. vseh sil} = \vec{N}(O)$$

$$\vec{N}(O) + (O-P_0) \times \vec{F} = \sum_{i=1}^N (P_i-O) \times \vec{F}_i + (O-P_0) \times \sum_{i=1}^N \vec{F}_i =$$

$$\parallel - \sum_{i=1}^N ((P_i-O) + (O-P_0)) \times \vec{F}_i = \sum (P_i-P_0) \times \vec{F}_i =$$

$$= \vec{N}(P_0)$$

$$\boxed{\frac{d\vec{L}(P_0)}{dt} = \vec{N}(P_0) + (P_0-P_*) \times m \ddot{\vec{P}}_0} \quad \text{odtud urt. količine relativnega gibanja}$$

$$\textcircled{1} \quad \underline{P_0 = P_*} : \quad \frac{d\vec{L}(P_0)}{dt} = \vec{N}(P_0)$$

$$\textcircled{2} \quad \underline{\ddot{\vec{P}}_0 = \vec{0}} : \quad \frac{d\vec{L}(P_0)}{dt} = \vec{N}(P_0)$$

$\Leftrightarrow$ brez o vrtilni količine
relativnega gibanja
med inercialnimi
koordinatnimi sistemi

$$\frac{d\vec{L}(O)}{dt} = \vec{N}(O) \quad ; \quad \left(\dot{\vec{P}}_* \mapsto \vec{c} + Q \dot{\vec{P}}_i \right. \\ \left. \text{hitrost mi invarijantna} \right)$$

Izrek Brez o vrtilni količini sistema svojo obliko pri $P_0 = P_*$ in pri $\ddot{\vec{P}}_0 = \vec{0}$.

Energija

$$m\ddot{\vec{P}} = \vec{F} \quad | \cdot \dot{\vec{P}}$$

$$\frac{d}{dt} \left(\frac{1}{2} m |\dot{\vec{P}}|^2 \right) = \vec{F} \cdot \dot{\vec{P}} = - \frac{d}{dt} U$$

$$\text{Newt. enačba: } m_i \ddot{\vec{P}}_i = \vec{F}_i + \sum_{\substack{j=1 \\ j \neq i}}^N \vec{F}_{ji} \quad | \cdot \dot{\vec{P}}_i \quad \left(\vec{F}_{ii} = 0 \right)$$

$$\boxed{T = \frac{1}{2} \sum_{i=1}^N m_i |\dot{\vec{P}}_i|^2} \quad \frac{d}{dt} \left(\frac{m_i |\dot{\vec{P}}_i|^2}{2} \right) = \vec{F}_i \cdot \dot{\vec{P}}_i + \sum_{j=1}^N \vec{F}_{ji} \cdot \dot{\vec{P}}_i$$

kinetična energ. sistema

$$\frac{dT}{dt} = \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{P}}_i + \sum_{ij} \vec{F}_{ji} \cdot \dot{\vec{P}}_i$$

moč zunanje sile moč notranje sile

$$\vec{F}_i = -\text{grad} U_i \quad ; \quad U_i = U_i(\vec{P}_i)$$

$$U = \sum U_i$$

$$\frac{dU}{dt} = \sum_{i=1}^N \frac{d}{dt} U_i(\vec{P}_i(t)) = \sum_{i=1}^N (\text{grad} U_i) \cdot \dot{\vec{P}}_i$$

moč zun. sile: $\sum \vec{F}_i \cdot \dot{\vec{P}}_i = -\frac{dU}{dt}$ (če so sile konzervativne)

moč notr. sile: $\vec{F}_{ji} = F_{ji} \frac{(\vec{P}_i - \vec{P}_j)}{|\vec{P}_i - \vec{P}_j|}$

$$\frac{\vec{r}}{r} = \frac{\vec{r}}{|\vec{r}|} = \text{grad} r \quad \left(\text{grad} r = \frac{\partial r}{\partial x} \vec{i} + \frac{\partial r}{\partial y} \vec{j} + \frac{\partial r}{\partial z} \vec{k} = \right.$$

$$\left. \Rightarrow \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r} \right.$$

$\Rightarrow$ notopazajo simetrično $\Rightarrow \text{grad} r = \frac{\vec{r}}{r}$

$$U_{ji}(\vec{P}_i, \vec{P}_j) = \int \vec{F}_{ji}(\vec{P}) d\vec{P} + C_{ji}$$

$$U_{ji} = U_{ji}(|\vec{P}_i - \vec{P}_j|) = U_{ji}$$

$\vec{F}_{ji} = \vec{F}_{ij}$

$$\frac{\partial}{\partial \vec{P}_i} U_{ji} = \vec{F}_{ji}(|\vec{P}_i - \vec{P}_j|) \cdot \frac{\partial}{\partial \vec{P}_i} |\vec{P}_i - \vec{P}_j|$$

$$\frac{\partial}{\partial \vec{P}_i} |\vec{P}_i - \vec{P}_j| = \frac{\partial |\vec{P}_i - \vec{P}_j|}{\partial (\vec{P}_i - \vec{P}_j)} \cdot \frac{\partial (\vec{P}_i - \vec{P}_j)}{\partial \vec{P}_i}$$

$$\frac{\vec{P}_i - \vec{P}_j}{|\vec{P}_i - \vec{P}_j|} \quad \vec{I}$$

U_{ji} je potencial notranje sile $\vec{F}_{ji}$.

$$\vec{F}_{ji} = -\left(\frac{\partial U_{ji}}{\partial \vec{P}_i}\right)^T = -\text{grad}_{\vec{P}_i} U_{ji}$$

$$\sum_{ij} \vec{F}_{ji} \cdot \dot{\vec{P}}_i = \frac{1}{2} \left(\sum_{ij} \vec{F}_{ji} \cdot \dot{\vec{P}}_i + \sum_{ij} \vec{F}_{ij} \cdot \dot{\vec{P}}_j \right)$$

$$\frac{d}{dt} U_i(|\vec{P}_i(t) - \vec{P}_j(t)|) = \left(\frac{\partial U_{ji}}{\partial \vec{P}_i}\right)^T \cdot \dot{\vec{P}}_i + \left(\frac{\partial U_{ji}}{\partial \vec{P}_j}\right)^T \cdot \dot{\vec{P}}_j = -\vec{F}_{ji} \cdot \dot{\vec{P}}_i - \vec{F}_{ij} \cdot \dot{\vec{P}}_j$$

$$\Rightarrow = -\frac{1}{2} \sum \frac{dU_{ji}}{dt} = \frac{d}{dt} \left(-\sum_{j < i} U_{ji} \right)$$

TRDITEV Če so notranje sile centralne, so tudi konzervativne.

Njihov potencial je U_{ji} , celotni potencial pa $\sum_{j < i} U_{ji} = U_n$

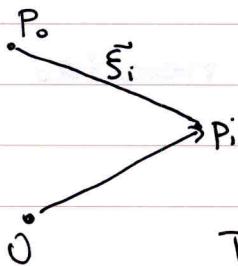
↓
potencial notranjih sile.

$$\left(\begin{array}{l} U_{ji} \text{ je potencial notranje sile } \vec{F}_{ji} \\ \vec{F}_{ji} = - \left(\frac{\partial U_{ji}}{\partial \vec{r}_i} \right)^T = - \text{grad}_{\vec{r}_i} U_{ji} \end{array} \right)$$

Energijska enačba za sistem materialnih točk:

$$\frac{d}{dt}(T + U_n) = \sum_{i=1}^N \vec{F}_i \cdot \dot{\vec{r}}_i$$

Če so zunanje sile konzervativne, je vsota $T + U_n + U$ konstanta gibanja



$$\vec{P}_i - \vec{O} = (\vec{P}_0 - \vec{O}) + \vec{\xi}_i$$

$$\dot{\vec{P}}_i = \dot{\vec{P}}_0 + \dot{\vec{\xi}}_i$$

$$T = \frac{1}{2} \sum_{i=1}^N m_i \dot{\vec{P}}_i \cdot \dot{\vec{P}}_i = \frac{1}{2} \sum m_i |\dot{\vec{P}}_0|^2 + \frac{1}{2} \sum m_i |\dot{\vec{\xi}}_i|^2 + \sum m_i \dot{\vec{P}}_0 \cdot \dot{\vec{\xi}}_i$$

kinetična energ.
gibanja glede na $\vec{P}_0$
= relativna kin. energija

$$\dot{\vec{\xi}}_i = \dot{\vec{P}}_i - \dot{\vec{P}}_0$$

$$T = \frac{1}{2} \sum m_i |\dot{\vec{\xi}}_i|^2 - \frac{1}{2} m |\dot{\vec{P}}_0|^2 \left(\sum m_i \dot{\vec{P}}_i \right) \cdot \dot{\vec{P}}_0$$

$$T = \frac{1}{2} \sum m_i |\dot{\vec{\xi}}_i|^2 + m_* \dot{\vec{P}}_* \cdot \dot{\vec{P}}_0 - \frac{1}{2} m |\dot{\vec{P}}_0|^2$$

$$\vec{P}_0 = \vec{P}_*$$

$$T = \frac{1}{2} \sum_{i=1}^N m_i |\dot{\vec{\xi}}_i|^2 + \frac{1}{2} m |\dot{\vec{P}}_*|^2$$

ZAPRTI SISTEM $\Leftrightarrow \vec{F}_i = \vec{0} \quad \forall i$

$$m \dot{\vec{P}}_* = \vec{0} \Rightarrow m \dot{\vec{P}}_* = \text{konst} = \vec{P}, \quad \vec{P}_* = \frac{1}{m} \vec{P} \cdot t + \vec{P}_*(t=0)$$

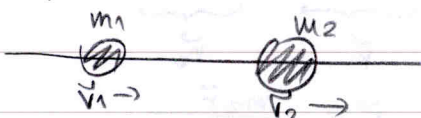
V zaprtem sistemu je gibalna kol. konstanta gibanja

$$\vec{P}, \vec{P}_*(t=0)$$

$$\frac{d\vec{L}(t)}{dt} = \vec{0} \Rightarrow \vec{L} = \text{konst}$$

$$T + U_n = \text{konst} \quad [E \text{ konst}]$$

Zaprta sistem ima skupaj 10 konstant.



pred trkom

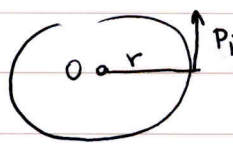


po trku

Neglede na vsto tika se gibalna količina drnanja

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

elastični trk: $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$



$$(P_i - O) \times m \dot{P}_i = r^2 \omega \vec{k}$$

$$\dot{P}_i = r \cdot \omega ; r^2 \omega = \text{konst}$$

Problem dveh teles

= dve telesi, ki tvorita zaprt sistem + sila med točkama je gravitacijska

Izberemo opazovalca z izhodiščem v masnem središču

$$m_1 \ddot{P}_1 = -G \frac{m_1 m_2}{|P_1 - P_2|^2} \cdot \frac{P_1 - P_2}{|P_1 - P_2|}$$

$$m_2 \ddot{P}_2 = -G \frac{m_1 m_2}{|P_2 - P_1|^2} \cdot \frac{P_2 - P_1}{|P_2 - P_1|}$$

$$\Rightarrow m_1 \ddot{P}_1 + m_2 \ddot{P}_2 = \vec{0} \quad \boxed{m \ddot{P}_* = 0}$$

$$\ddot{P}_1 - \ddot{P}_2 = -G \frac{1}{(P_1 - P_2)^3} \left((P_1 - P_2) m_2 - (P_2 - P_1) m_1 \right)$$

$$\vec{r} = P_1 - P_2$$

$$\ddot{\vec{r}} = -G \frac{1}{|\vec{r}|^3} (m_2 \vec{r} + m_1 \vec{r})$$

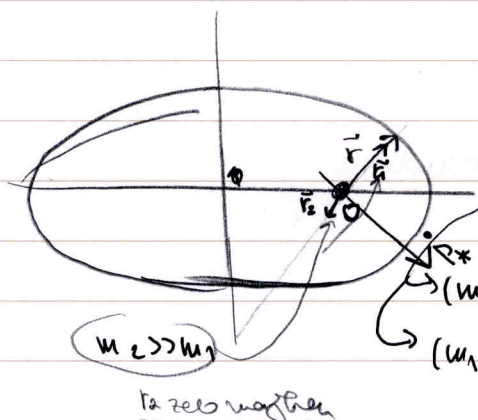
$$\ddot{\vec{r}} = -G \frac{m_1 + m_2}{|\vec{r}|^3} \cdot \vec{r}$$

$$\frac{m_1 m_2}{m_1 + m_2} \ddot{\vec{r}} = -G \frac{m_1 \cdot m_2}{|\vec{r}|^2} \cdot \frac{\vec{r}}{|\vec{r}|}$$

grav. sila med dvema masama

μ reducirana masa

Gibanje mrt točke z reducirano maso v polju gravitacijske sile s centrom sile v koord. središču



$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

$$\vec{r}_1 - \vec{r}_2 = \vec{r}$$

$$\vec{r} = P_1 - P_2 = \underbrace{(P_1 - O)}_{\vec{r}_1} - \underbrace{(P_2 - O)}_{\vec{r}_2}$$

$$(m_1 + m_2) \vec{r}_1 = m_2 \vec{r} \Rightarrow \vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2}$$

$$(m_1 + m_2) \vec{r}_2 = -m_1 \vec{r} \Rightarrow \vec{r}_2 = -\frac{m_1 \vec{r}}{m_1 + m_2}$$

$$\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2}$$

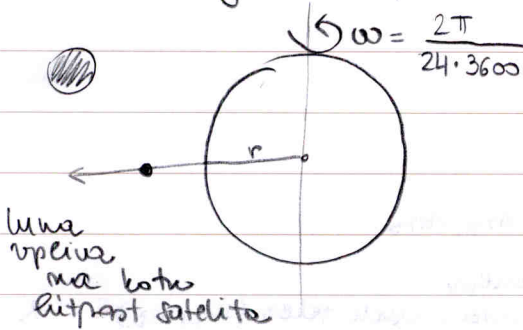
$$\vec{r}_2 = -\frac{m_1 \vec{r}}{m_1 + m_2}$$

$m_2 \gg m_1$

iz zelo majhnih

Problem treh teles

(sonce-zemlja-luna, satelit-zemlja-sonce ...)



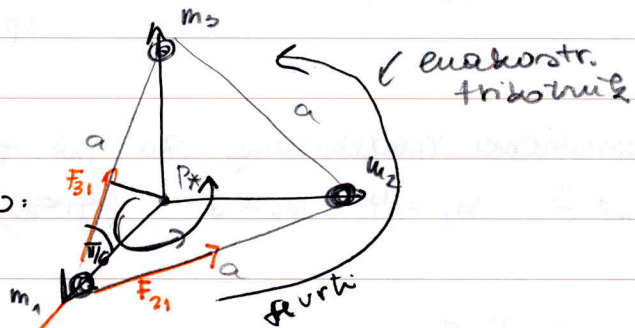
Splosne resitve ni, ji pa konstantna konfiguracija

Konstantna konfiguracija:

Dokaz za $m_1 = m_2 = m_3$:

Privzamemo, da je enakokr. Δ in iščemo ω :

$$\vec{0} = \vec{F}_{21} + \vec{F}_{31} + (-m \cdot \vec{\omega} \times (\vec{\omega} \times (\vec{P}_1 - \vec{P}_*)))$$



$$m \omega^2 r = 2 \cdot m \cdot \frac{1}{a^2} \cos \pi/6 = 2m \cdot \frac{1}{a^2} \sqrt{3}$$

$$= m \omega^2 \frac{a}{\sqrt{3}}$$

$$\cos \pi/6 = \frac{a}{2r} \Rightarrow r = \frac{a}{2 \cos \pi/6}$$

$$r = \frac{a}{\sqrt{3}}$$

$$\omega^2 = \frac{3m}{a^3}$$

$$a = \sqrt[3]{\frac{3m}{\omega^2}}$$

splosno: $a = \sqrt[3]{\frac{(m_1 + m_2 + m_3) \cdot r^3}{\omega^2}}$

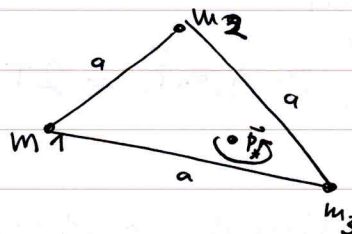
za 3 enake mase
Konstantna konfiguracija je nestabilna

Če 3 neenake mase:

Ta je stabilna za

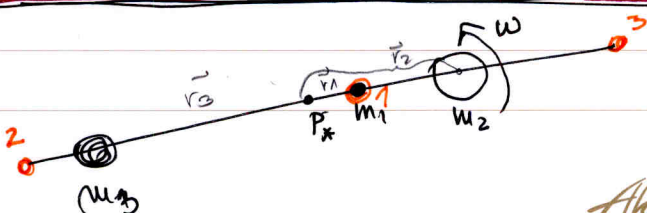
$$(m_2 + m_3)^2 > 2m_1 m_2 m_3$$

$$m_1 \ll 1$$



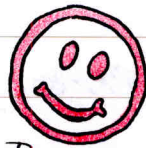
$$m_1 < m_2 < m_3$$

Či so vsa 3 telesa na isti premici:



• 3 možne resitve

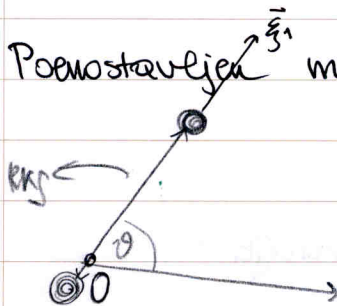
$$m_1 \omega^2 r_1 = \cancel{8} \cancel{m_1} m_3 \frac{1}{(r_1 + r_3)^2} \neq \cancel{8} m_2 \cancel{m_1} \frac{1}{(r_2 - r_1)^2}$$



$$\omega^2 r_1 (r_1 + r_3)^2 (r_2 - r_1)^2 - 8 m_3 (r_2 - r_1)^2 + 8 m_2 (r_1 + r_3)^2 = 0$$

3 realne rešitve

Konfiguracija je nestabilna.



Poenostavljen model treh teles: $m_1 \ll m_2, m_3$

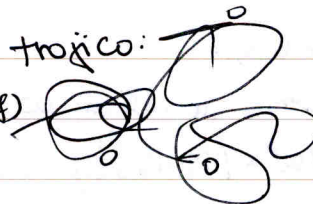
$\uparrow$ zanemarljiv
 $\Rightarrow$ rešimo sistem dveh teles (m_2, m_3)

Naloga: ~~raziskati~~ gibanje mat. točke pod vplivom sil z delnico centra mase.

Numerično modeliranje za pit. trojico:

$$m_1 = 3 \quad m_2 = 4 \quad m_3 = 5$$

(graf)



$$\vec{L} = \vec{L}(P_*)$$

Če je vektor vrtilne kol. sistema ~~treh~~ teles enak nič, je gibanje pravokotno.

Če je $\vec{L} \neq \vec{0}$, potem obratni tok vseh treh teles ni možen.

VIRIAL

$$W(t) = \sum_{i=1}^N (P_i - O) \cdot m_i \dot{P}_i$$

$$\frac{\partial W}{\partial t} = \underbrace{\sum_{i=1}^N m_i \dot{P}_i \dot{P}_i}_{2T} + \sum_{i=1}^N (P_i - O) m_i \ddot{P}_i = 2T + \sum (P_i - O) \vec{F}_i + \sum_{ij} (P_i - O) \vec{F}_{ji}$$

$$\begin{aligned} \sum_{ij} (P_i - O) \vec{F}_{ji} &= \frac{1}{2} \left(\sum_{ij} (P_i - O) \vec{F}_{ji} + \sum_{ji} (P_j - O) \vec{F}_{ij} \right) = \frac{1}{2} \sum_{ij} ((P_i - O) - (P_j - O)) \vec{F}_{ji} = \\ &= \frac{1}{2} \sum_{ij} (P_i - P_j) \vec{F}_{ji} = \sum_{i < j} (P_i - P_j) \vec{F}_{ji} \end{aligned}$$

$$\vec{F}_{ji} = \left(\frac{\partial U_{ji}}{\partial P_i} \right)^T$$

$$U_{ji} = \int_{P_j}^{P_i} \vec{F}_{ji}(r) dr \Rightarrow \vec{F}_{ji} = \left(\frac{\partial U_{ji}}{\partial \mathbf{r}} \right)^T = \left(\frac{\partial U_{ji}}{\partial (P_i - P_j)} \cdot \frac{\partial (P_i - P_j)}{\partial P_i} \right)^T = \left(\frac{\partial U_{ji}}{\partial (P_i - P_j)} \right)^T$$

$$\sum_{i,j} (P_i - P_j) \vec{F}_{ji} = \sum_{i,j} (P_i - P_j) \frac{\partial U_{ji}}{\partial (P_i - P_j)}.$$

Def: f je homogena funkcija reda p , če je $f(\lambda(P-O)) = \lambda^p f(P-O)$.
 Odvajamo po λ : $\text{grad } f|_{(\lambda(P-O))} \cdot (P-O) = p \lambda^{p-1} f(P-O)$

$$\lambda = 1 \Rightarrow (\text{grad } f)(P-O) = p f(P-O). \quad * \text{ Eulerjev izrek o homogeni funkciji}$$

$f(P-O)$ je homogena $\Leftrightarrow *$

$$1) \vec{F}_{ji} = - \frac{m_1 m_2}{|P_i - P_j|^2} \frac{P_i - P_j}{|P_i - P_j|}$$

$$U_{ji} = - \gamma \frac{1}{|P_i - P_j|}; \quad \gamma = \gamma m_1 m_2$$

Potencial gravitacijske sile je homogena funkcija reda $p = -1$.

$$2) \vec{F}_{ji} = -k |P_i - P_j| \frac{P_i - P_j}{|P_i - P_j|} = -k (P_i - P_j)$$

$$U_{ji} = \frac{1}{2} k |P_i - P_j|^2 \quad \text{Potencial Hookeove sile je hom. f. reda } p = 2.$$

TRDITEV Če je sistem zaprt in so potenciali notr. sil homogene f. reda p , je odvod vartala po času

$$\frac{dW}{dt} = 2T - p \cdot U; \quad U = \sum_{j,i} U_{ji}$$

U potencial not. sil, ki je homogena f. reda p .

Povprečna vrednost (časovno povprečje)

$$\langle f \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T+T} f(\tau) d\tau$$

$$f \text{ omejena} \Rightarrow |\langle f \rangle| \leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T+T} |f(\tau)| d\tau \leq M$$

$\|f\|_\infty = M.$

$$\left\langle \frac{dW}{dt} \right\rangle = \lim_{T \rightarrow \infty} \frac{1}{T} (W(t_0+T) - W(t_0))$$

$$W \text{ omejena} \Rightarrow \left\langle \frac{dW}{dt} \right\rangle = 0$$

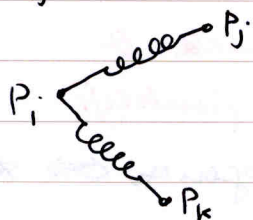
$$W = W(0) = \sum_{i=1}^N (P_i - 0) \cdot m_i \cdot \dot{P}_i$$

$$\text{omejeno gibanje } |P_i - 0| < M$$

$$\text{regularno gibanje } |\dot{P}_i| < M$$

$$\text{omejeno reg. gibanje} \Rightarrow \langle W \rangle = 0.$$

Uj: Hookovi potenciali, $p=2$



Pojubni sistem lahko v okolici ravnovesne lege aproksimiramo s Hook. potenciali.

V okolici ravnovesne lege imamo omejeno

regul. gibanje:

$$\left\langle \frac{dW}{dt} \right\rangle = 0 \Rightarrow \langle 2T - 2U \rangle = 0$$

$$\Rightarrow \langle T \rangle = \langle U \rangle$$

$$E = T + U \Rightarrow \langle E \rangle = E = \langle T \rangle + \langle U \rangle$$

$$\langle T \rangle = \langle U \rangle = \frac{1}{2} E$$

ekviparticijska energija
(statistična mehanika)

Problem N teles

$$p = -1$$

$$\left\langle \frac{dW}{dt} \right\rangle = 2\langle T \rangle + \langle U \rangle = E + \langle T \rangle ; E = \langle T \rangle + \langle U \rangle$$

$$\text{omejeno regularno gibanje } \left\langle \frac{dW}{dt} \right\rangle = 0$$

$$\Rightarrow E = -\langle T \rangle$$

$E > 0 \Rightarrow$ vsaj ena točka ubreži v neskončnost

$$J(0) = \frac{1}{2} \sum_{i=1}^N m_i |P_i - 0|^2 \quad \text{vztrajnostni moment sistema}$$

$$\frac{dJ}{dt} = \sum_{i=1}^N (P_i - 0) m_i \dot{P}_i = W$$

$$\frac{d^2 J}{dt^2} = \frac{dW}{dt} = 2T - pU \quad \downarrow p=-1$$

$$= 2T + U = E + T \geq E$$

$$\frac{dJ}{dt} \geq E \cdot t + C_1$$

$$J \geq \frac{1}{2} E t^2 + C_1 t + C_2 ;$$

$$E > 0 \Rightarrow \lim_{t \rightarrow \infty} J = \infty$$

Vsaj ena točka ubreži
v neskončnost
Vf mora postati ∞
 $\Rightarrow$ vsaj ena točka ubreži
v neskončnost