5. NEKAJ UPORAB LINEARNE ALGEBRE V DISKRETNI MATEMATIKI

Izrek. (**Fisherjeva neenakost**) Naj bo B 2-načrt s parametri (v, k, λ_2) in naj bo $v > k$. Tedaj je $b \geq v$.

Dokaz. Elementi načrta : $x_1, x_2, \dots, x_v$

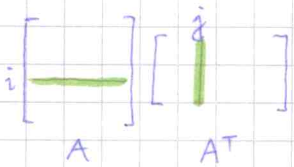
Bloki načrta : $B_1, B_2, \dots, B_b$.

Incidenčna matrika načrta A : $(A)_{ij} = \begin{cases} 1 ; & x_i \in B_j \\ 0 ; & \text{sicer} \end{cases}$

A je $v \times b$ matrika

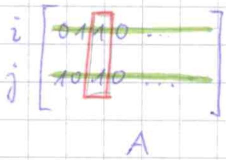
$$\underset{\substack{\uparrow \\ v\times v}}{M} = \underset{\substack{\uparrow \\ v\times b}}{A}\,\underset{\substack{\uparrow \\ b\times v}}{A^T}$$

$(M)_{ij} =$



= skalarni produkt i-te z j-to vrstico matrike A

=



= # stolpcev, v katerih je = 1 v obeh vrsticah

= # blokov, v katerih sta tako x_i kot x_j

$= \lambda_2 \quad \forall i, j \,,\ i \neq j$

$(M)_{ii}$ = # enic v i-ti vrstici matrike A $= \lambda_1 = \lambda_2 \frac{v-1}{k-1} =: r$

$$\underset{v\times v}{M} = \begin{bmatrix} r & \lambda_2 & \cdots & \lambda_2 \\ \lambda_2 & r & & \vdots \\ \vdots & & \ddots & \lambda_2 \\ \lambda_2 & \cdots & \lambda_2 & r \end{bmatrix} \qquad \det M = ?$$

$$M \sim \begin{bmatrix} r-\lambda_2 & 0 & \cdots & 0 & \lambda_2 - r \\ 0 & r-\lambda_2 & 0 & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & \lambda_2 - r \\ 0 & \cdots & 0 & r-\lambda_2 & \lambda_2 - r \\ \lambda_2 & \lambda_2 & \cdots & \lambda_2 & r \end{bmatrix} \sim \begin{bmatrix} r-\lambda_2 & 0 & \cdots & \cdots & 0 \\ 0 & r-\lambda_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & r-\lambda_2 & 0 \\ \lambda_2 & \cdots & \cdots & \lambda_2 & (v-1)\lambda_2 + r \end{bmatrix}$$

$$\det M = (\underbrace{(v-1)}_{>0}\underset{\overset{\vee}{0}}{\lambda_2} + \underset{\overset{\vee}{0}}{r})(\underbrace{r-\lambda_2}_{>0})^{v-1} \qquad r = \lambda_2 \underbrace{\frac{v-1}{k-1}}_{>1} > \lambda_2 \Rightarrow r - \lambda_2 > 0$$

$\det M > 0 \Rightarrow$ M je obrnljiva matrika $\Leftrightarrow$ rang$(M) = v$.

Recimo, da je $b < v$. $\underset{v \times b}{A} \Rightarrow$ rang$(A) \leq b$

rang$(A^T) \leq b$

$$\text{rang}(AA^T) \leq \min\{\text{rang}(A), \text{rang}(A^T)\} \leq b < v = \text{rang}(\overbrace{AA^T}^{M}) \qquad \rightarrow\leftarrow \qquad \blacksquare$$

Zgled: $b = v$ ... sodo število

(v, k, λ_2) 2-načrt

$k - \lambda_2 = a^2$ (popoln kvadrat)

$$k = \lambda_1 = \lambda_2 \frac{v-1}{k-1} \Rightarrow k(k-1) = \lambda_2(v-1)$$

$$\det(AA^T) = (k(k-1)+k)(k-\lambda_2)^{v-1} = k^2 \underbrace{(k-\lambda_2)^{v-1}}_{\text{pop. kvadrat}}$$

$$\overset{\|}{(\det A)^2} \qquad |\det A| = k(k-\lambda_2)^{\frac{v-1}{2}}$$

Ker $v-1 \in L \Rightarrow k - \lambda_2$ je pop. kvadrat.

5.2. POKRITJA S POLNIMI DVODELNIMI GRAFI

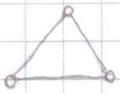
Problem: Polni graf K_n želimo pokriti s po povezavah disjunktnimi polnimi dvodelnimi grafi. Določiti moramo najmanjše število takih polnih dvodelnih grafov.

Oznaka: $m(n)$.

Zgledi: K_2:

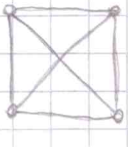
$K_{1,1}$ $\quad m(2) = 1$

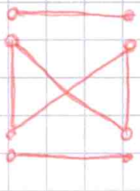
K_3:



$K_{1,2}$ $K_{1,1}$ $\quad m(3) = 2$

K_4:

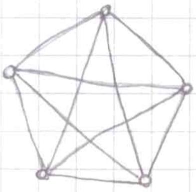


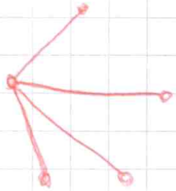


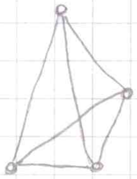
$K_{1,1}$ $K_{1,1}$ $K_{2,2}$

$m(4) = 3$

K_5 :







$m(5) = 1 + m(4) = 4$

$\boxed{m(n) \leq n-1}$

Pokrijemo K_n s $K_{1,n-1}$. Ostane K_{n-1}. (indukcija)

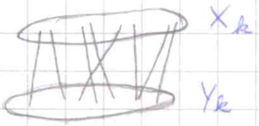
Izrek. $m(n) = n-1$.

Dokaz. $m(n) \geq n-1$

$V(K_n) = \{1, 2, \ldots, n\}$.

Naj bo $B_1, B_2, \ldots, B_m$ pokritje K_n, kjer so B_i polni dvodelni grafi. Naj bo $B_k = X_k + Y_k$ dvodelno razbitje od B_k:

B_k :



$B_k \longmapsto A_k$
$n \times n$ matrika

$$(A_k)_{ij} = \begin{cases} 1; & i \in X_k,\ j \in Y_k \\ 0; & \text{sicer} \end{cases}$$

$A := A_1 + A_2 + \ldots + A_m$

$(A)_{ii} = 0$ za $1 \leq i \leq n$

$i \neq j$: $(A)_{ij} = 1 \iff (A)_{ji} = 0$. Res, saj smo K_n pokrili z disjunktnimi (po povezavah) grafi.

$$J_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \qquad A + A^T = J_n - I_n \qquad I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Trdimo: $\text{rang}(A) \geq n-1$.

Če to dokažemo, tedaj: $n-1 \leq \text{rang}(A) = \text{rang}(A_1 + A_2 + \ldots + A_m) \leq \text{rang}(A_1) + \ldots + \text{rang}(A_m) = m$.

Recimo nasprotno: $\text{rang}(A) \leq n-2$.
(s protislovjem)

$$\underset{(n+1)\times n}{\tilde{A}} = \begin{bmatrix} \boxed{\quad A \quad} \\ 1\ 1\ \cdots\ 1 \end{bmatrix}$$

rang($\tilde{A}$) $\leq m-1$, torej so stolpci v $\tilde{A}$ ($\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_m$) lin. odvisni: $\exists\, \alpha_1, \alpha_2, \dots, \alpha_m$, ne vsi 0 $\ni$:

$$\alpha_1 \tilde{A}_1 + \alpha_2 \tilde{A}_2 + \dots + \alpha_m \tilde{A}_m = 0$$

$$x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \neq 0 \qquad \tilde{A}x = \begin{bmatrix} a_{11}\alpha_1 + \dots + a_{1m}\alpha_m \\ \\ a_{m1}\alpha_1 + \dots + a_{mm}\alpha_m \\ \alpha_1 + \alpha_2 + \dots + \alpha_m \end{bmatrix} = 0$$

$$\Rightarrow Ax = 0, \ \sum_{i=1}^{m} \alpha_i = 0 \Rightarrow J_m x = 0$$

$$x^T(A+A^T)x = x^TAx + x^TA^Tx = x^T(\underbrace{Ax}_{=0}) + (\underbrace{Ax}_{=0})^T x = 0$$

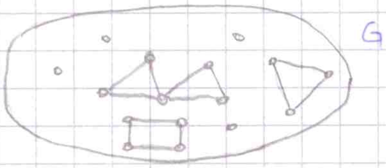
$$x^T(A+A^T)x = x^T(J_m - I_m)x = x^T \underbrace{J_m x}_{0} - x^T I_m x = -x^Tx =$$

$$= -\|x\|^2 \underset{x \neq 0}{<} 0 \qquad \rightarrow\leftarrow$$

■

5.3. PROSTORI CIKLOV, KROŽENJ IN PREREZOV

Def. Naj bo G graf. Tedaj je množica povezav $E' \subseteq E(G)$ soda množica, če ima vsako vozlišče grafa $(V(G), E')$ sodo stopnjo.



Primeri:
- $\emptyset$
- cikli kot podgrafi (cikel $\equiv$ množica njegovih povezav).
- K_{2n+1} (kot podgraf)

Trditev. Množica E' je soda natanko tedaj, ko obstajajo po povezavah disjunktni cikli $E_1, E_2, \dots, E_k$, tako da je

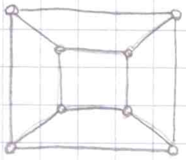
$$E' = E_1 \cup E_2 \cup \dots \cup E_k.$$

Dokaz. ($\Rightarrow$) Naj bo E' soda množica povezav. Če so vse stopnje 0, ni kaj dokazovati.

Sicer imamo cikel v E'. Po njegovi odstranitvi spet dobimo sodo množico. Nato uporabimo indukcijo.

$(\Leftarrow)$: očitno

Zgled. Poišči vse sode množice Q_3.

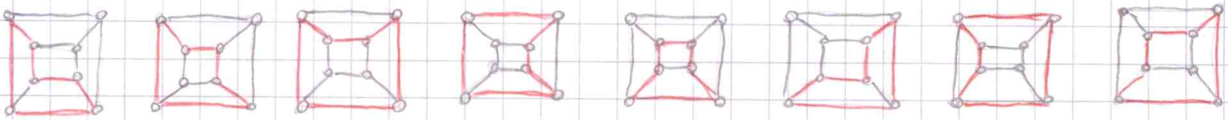


- $\emptyset$
- $6\times\ C_4$
- $3\times\ C_4 \cup C_4$
- $16\times\ C_6$

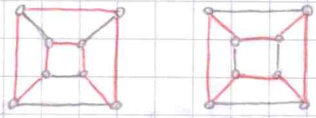
#povezav ↓ vsaka pot. se pojavi v 8 ciklih

$$\frac{12\cdot 8}{6}$$

← vsak cikel smo šteli 6x



- $6\times\ C_8$



vseh sodih množic: 32

$G=(V(G),E(G))$

$E(G) \dots\ e_1, e_2, \dots, e_m$ urejena množica povezav

$E' \subseteq E(G) \longmapsto v_{E'} \in \{0,1\}^m$

$$v_{E'}(i) = \begin{cases} 1; & e_i \in E' \\ 0; & \text{sicer} \end{cases}$$

$\mathcal{E} = \{v_A \mid A \text{ soda množica}\}$; $A \subseteq V(G)$

Označimo tudi: $\mathcal{E}(G)$

$v_\emptyset = (0,0,\dots,0)$

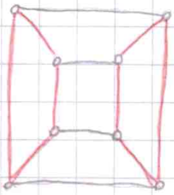
$+ : \mathcal{E}\times\mathcal{E} \to \mathcal{E}$

$v_A + v_B = v_{A\oplus B}$ ← simetrična razlika $A\oplus B = (A\cup B)\setminus(A\cap B)$

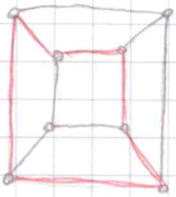
Če sta A, B sodi $\Rightarrow A\oplus B$ soda.

Primer:

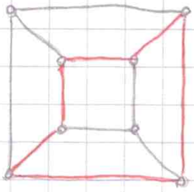
A:



B:



$A\oplus B$:



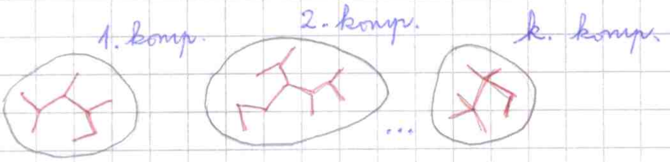
$\mathbb{Z}_2$ (obseg) : $0 \cdot v_A = 0$

$1 \cdot v_A = v_A$

Izrek. Naj bo G graf s k komponentami, n vozlišči in m povezavami. Tedaj $(\mathcal{E}, +, \cdot)$ tvori vektorski prostor dimenzije $m-n+k$. ($\dim(\mathcal{E}(G)) = |E(G)| - |V(G)| + (\#\text{komponent})$)

Skica dokaza. operacija + je notranja, ostali aksiomi za vekt. pr. so očitni (ko opazimo, da + predstavlja seštevanje po modulu 2).

Naj bo T vpet gozd grafa G:



Vsaka povezava iz $E(G) \setminus T$ določa natanko en cikel

C_e ... enoličen cikel, ki ga porodi $e \in E(G) \setminus T$

$X = \{ v_{C_e} \mid e \in E(G) \setminus T \}$ je baza

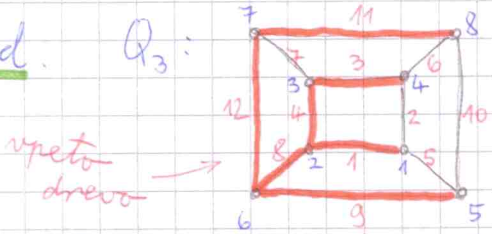
neodvisnost: $e_i \in E(G) \setminus T$:

$$v_{C_{e_i}} = (\ldots, \underset{\substack{\uparrow \\ i\text{-to mesto}}}{1}, \ldots)$$

in noben drug vektor iz X nima na tem mestu 1. □

Opomba. $m-n+k$ imenujemo ciklomatično št. grafa.

Zgled. Q_3 :



$\dim(Q_3) = 12 - 8 + 1 = 5$

		cikli	karak. vektorji ciklov
Baza:	14 :	1, 4, 3, 2	(1, 1, 1, 1, 0, 0, 0, 0)
	15 :	1, 5, 6, 2	(1 1, 0, 0, 1, 1, 0, 0)
	37 :	3, 7, 6, 2	(0, 1, 1, 0, 0, 1, 1, 0)
	48 :	4, 8, 7, 6, 2, 3	(0, 1, 1, 1, 0, 1, 1, 1)
	58 :	5, 8, 7, 6	(0, 0, 0, 0, 1, 1, 1, 1)

↖ jih ne potrebujemo (nimajo smisla)

bazni vektorji (iz $\mathcal{E}$):

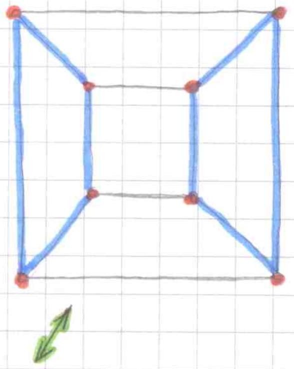
$v_2 = (1,1,1,1,0,0,0,0,0,0,0,0)$

$v_5 = (1,0,0,0,1,0,0,1,1,0,0,0)$

$v_7 = (0,0,0,1,0,0,1,1,0,0,0,1)$

$v_6 = (0,0,1,1,0,1,0,1,0,0,1,1)$

$v_{10} = (0,0,0,0,0,0,0,0,1,1,1,1)$



$v_2 + v_5 + v_6 + v_7 + v_{10} = (0,1,0,1,1,1,1,1,0,1,0,1)$

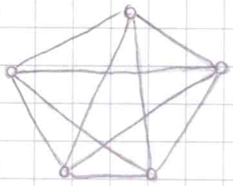
Posledica. Graf G z n vozlišči, m povezavami in k komponentami vsebuje 2^{m-n+k} sodih množic.

Dokaz. $\dim(\mathcal{E}(G)) = m-n+k$. Vsaka lin. kombinacija baznih vektorjev nam da eno sodo množico. Linearnih kombinacij je 2^{m-n+k}:

$v_1, \dots, v_{m-n+k}$ bazni vektorji: $\sum_{i=1}^{m-n+k} \alpha_i v_i \quad (\alpha_i \in \mathbb{Z}_2)$

in ta prireditev je injektivna, saj je razvoj po bazi enoličen. (če $\sum \alpha_i v_i = \sum \beta_i v_i \Rightarrow \alpha_i = \beta_i \ \forall i$)

Zgled. $G = K_5$. Poišči vse sode množice.

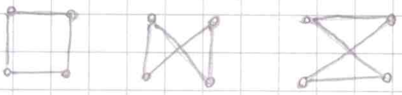


$n = 5, \ m = 10, \ k = 1$

$\Rightarrow \dim(\mathcal{E}(G)) = 10-5+1 = 6$

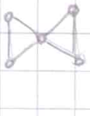
sodih množic: $2^6 = 64$

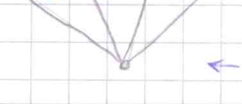
- trikotniki: $\binom{5}{3} = 10$
- štirikotniki (4-cikli): $\binom{5}{4} \cdot \underline{\underline{3}} = 15$



- 5-cikli: $\frac{5!}{5 \cdot \underline{\underline{2}}} = 12$

$\leftarrow$ vsak cikel smo šteli v obe smeri

-  :

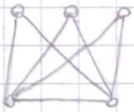


} 3 možnosti

$3 \cdot 5 = 15$

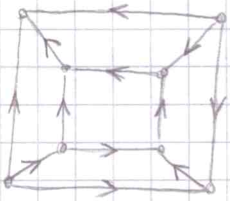
$\leftarrow$ 5 možnosti

- $\emptyset, K_5$: $1+1 = 2$

-  : 10 možnosti $\binom{5}{2}$, ker točke stopnje 2 enolično določajo ta podgraf (oz. točke stopnje 4)

PROSTOR KROŽENJ

$G = (V(G), E(G))$ graf. Digraf $\vec{G} = (V(G), \vec{E}(G))$, ki ga dobimo iz G z usmeritvijo vseh njegovih povezav, imenujemo **usmeritev od G**.

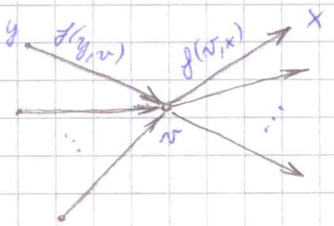


usmerjena povezava: (x, y)

Usmeritev $\vec{G}$ bo v nadaljevanju poljubna, a fiksna.

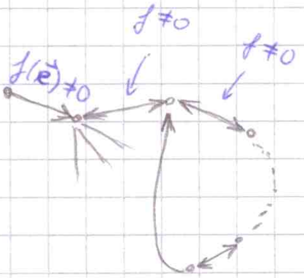
Preslikava $f: \vec{E}(G) \longrightarrow \mathbb{R}$ je **kroženje**, če za vsako vozlišče $v \in V(G)$ velja:

$$\sum_{\substack{x \in V(G) \\ (v,x) \in \vec{E}}} f((v,x)) = \sum_{\substack{x \in V(G) \\ (x,v) \in \vec{E}}} f((x,v)).$$

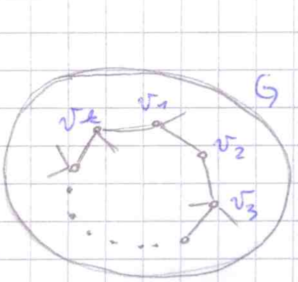


$f \equiv 0$ ... **ničelno kroženje**

Naj $\vec{G}$ premore neničelno kroženje. Torej, v tem primeru G premore cikel. (V drevesih ni neničelnih kroženj.)

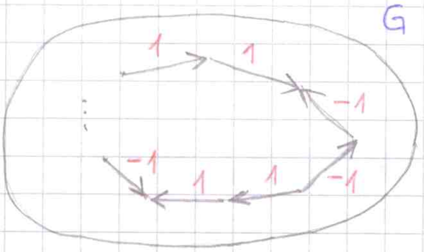


Naj bo G graf s ciklom $\vec{C}$: $v_1, v_2, \ldots, v_k, v_1$. $\vec{G} = (V, \vec{E})$. Definirajmo f takole:

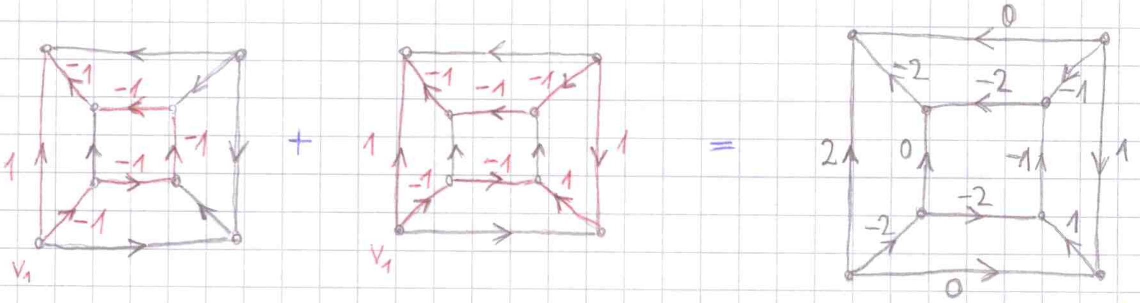


- če je $(v_i, v_{i+1}) \in \vec{E}$, naj bo $f(v_i, v_{i+1}) = 1$
- če je $(v_{i+1}, v_i) \in \vec{E}$, naj bo $f(v_{i+1}, v_i) = -1$
- če je $(x, y) \notin \vec{C}$, naj bo $f(x, y) = 0$

Tedaj je f kroženje:



Zgled.



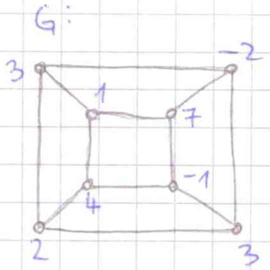
to je tudi kroženje

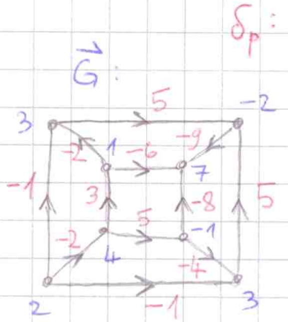
Naj bosta f in g kroženji, $\alpha, \beta \in \mathbb{R}$. $\alpha f + \beta g$ je tudi kroženje. Torej kroženja tvorijo realen vekt. pr. Njegova dimenzija je $m - n + k$, kjer je m # povezav, n # točk in k # komponent.

PROSTOR PREREZOV

$p : V(G) \longrightarrow \mathbb{R}$ ... potencial

$G \longmapsto \vec{G} = (V(G), \vec{E}(G))$





$\delta_p(x,y) = p(x) - p(y)$ ... razlika potenciala

$\uparrow (x,y) \in \vec{E}(G)$

Funkcija $g : \vec{E}(G) \longrightarrow \mathbb{R}$ je razlika potenciala, če obstaja $p : V(G) \longrightarrow \mathbb{R}$, tako da je $g(x,y) = p(x) - p(y)$.

δ_p, δ_q ... razliki potencialov p in q

$\delta_p + \delta_q$ je tudi razlika potencialov (za potencial $p + q$)

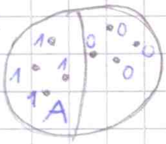
$\alpha \cdot \delta_p$ je tudi razlika potencialov (za potencial $\alpha \cdot p$)

Vektorskemu prostoru vseh razlik potencialov pravimo **prostor prerezov**; to je realen vektorski prostor.

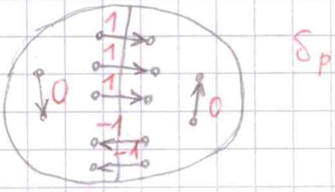
$A \subseteq V(G)$

$p : V(G) \to \{0, 1\}$ $\qquad p(u) = \begin{cases} 1 ; & u \in A \\ 0 ; & \text{sicer} \leftarrow V(G) \setminus A \end{cases}$

G:



$\vec{G}$:



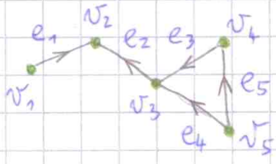
$V(G) = \{v_1, v_2, \dots, v_n\}$

$E(G) = \{e_1, \dots, e_m\}$ $\quad \vec{E}(G) = \{\vec{e}_1, \dots, \vec{e}_m\}$

Incidenčna matrika D je $n \times m$ matrika:

$$(D)_{ij} = \begin{cases} 1 ; & v_i \text{ je začetno krajišče od } \vec{e}_j \\ -1 ; & v_i \text{ je končno krajišče od } \vec{e}_j \\ 0 ; & \text{sicer} \end{cases}$$

Zgled.



$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

$\mathcal{C}$ ... prostor kroženj

$\mathcal{R}$ ... prostor prerezov

Izrek. Naj bo G poljuben graf. Tedaj je njegov prostor prerezov $\mathcal{R}$ generiran z vrsticami incidenčne matrike D. Nadalje je njegova dimenzija enaka $n-k$ in je prostor kroženj njegov ortogonalni komplement:

$$\mathcal{C} = \{ x \in R^m \mid x^T y = 0 \text{ za vse } y \in \mathcal{R} \}.$$

Dokaz. Vrstice matrike $\mathcal{D}$ pripadajo $\mathcal{R}$.

Za vse v_i naj bo $p_i(v_j) = \delta_{ij}$ $(1 \leq j \leq n)$.

$$\delta_{p_i}(e_j) = \begin{cases} 1; & \text{če je } v_i \text{ začetno krajišče } e_j \\ -1; & \text{če je } v_i \text{ končno krajišče } e_j \\ 0; & \text{sicer} \end{cases}$$

$\Rightarrow$ i-ta vrstica matrike $\mathcal{D}$ je torej natanko δ_{p_i}.

δ_p razlika potenciala $\Rightarrow$ δ_p je lin. kombinacija vrstic matrike $\mathcal{D}$

$$\delta_p(e_j) = p(v_r) - p(v_s) = \sum_{i=1}^{n} d_{ij}\, p(v_i)$$

$$\mathcal{D}:\ \begin{array}{r|ccc|l} & & d_{rj} & & \\ v_r: & & 1 & & \leftarrow r \\ & & d_{sj} & & \\ v_s: & & -1 & & \leftarrow s \\ \hline & & \uparrow & & \\ & & j & & \end{array}$$

$$\begin{bmatrix} \delta_p(e_1) \\ \vdots \\ \delta_p(e_m) \end{bmatrix} = \begin{bmatrix} d_{11}p(v_1) + \ldots + d_{n1}p(v_n) \\ \vdots \\ d_{1m}p(v_1) + \ldots + d_{n,m}p(v_n) \end{bmatrix}$$

poljubna razlika potenciala

$$= p(v_1)\begin{bmatrix} d_{11} \\ d_{12} \\ \vdots \\ d_{1m} \end{bmatrix} + p(v_2)\begin{bmatrix} d_{21} \\ d_{22} \\ \vdots \\ d_{2m} \end{bmatrix} + \ldots + p(v_n)\begin{bmatrix} d_{n1} \\ d_{n2} \\ \vdots \\ d_{n,m} \end{bmatrix}$$

1. vrstica $\mathcal{D}$ (kot vektor iz $\mathbb{R}^m$)

2. vrstica

n. vrstica

$\mathcal{C} = \mathcal{R}^{\perp}$ Naj bo f poljubno kroženje:

$$\forall v \in V(G): \sum_{\substack{x \in V(G) \\ (v,x) \in \vec{E}}} f(v,x) = \sum_{\substack{x \in V(G) \\ (x,v) \in \vec{E}}} f(x,v)$$ oziroma:

$$\sum_{\substack{x \in V(G) \\ (v,x) \in \vec{E}}} f(v,x) - \sum_{\substack{x \in V(G) \\ (x,v) \in \vec{E}}} f(x,v) = 0.$$ Naj bo $v = v_i$:

$$\sum_{j=1}^{m} f(\vec{e}_j) \cdot d_{ij} = 0.$$

$\Rightarrow$ f je $\perp$ na i-to vrstico matrike D. $(\forall i \in \{1,\dots,n\})$
Ker so vrstice generatorji prostora $\mathcal{R}$, je $f \perp$ na vse vektorje iz $\mathcal{R}$.

$\mathcal{R} \oplus \mathcal{R}^{\perp} = \mathcal{R} \oplus \mathcal{C} =$ "prostor vseh funkcij na povezavah"

$$\dim(\mathcal{R} \oplus \mathcal{C}) = m = \dim \mathcal{R} + \underbrace{\dim \mathcal{C}}_{m-n+k} \Rightarrow \dim \mathcal{R} = n-k.$$ ■

5.4. LASTNE VREDNOSTI/VEKTORJI IN UPORABE

G ... graf. $A(G)$ ... matrika sosednosti.

Def. Lastne vrednosti in lastni vektorji grafa G so lastne vrednosti in lastni vektorji njegove matrike sosednosti.

$A(G)$ realna simetrična matrika $\Rightarrow$ lastne vrednosti grafa so realne.

Velja tudi, da obstaja ON baza iz lastnih vektorjev (prostora $\mathbb{R}^n$).

Zgled. $G = K_n$. Izračunaj lastne vrednosti in lastne vektorje.

$$A(K_n) - \lambda I_n = \begin{bmatrix} -\lambda & 1 & 1 & 1 & \cdots \\ 1 & -\lambda & 1 & 1 & \cdots \\ 1 & 1 & -\lambda & 1 & \cdots \\ 1 & 1 & 1 & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\det(A(K_n) - \lambda I_n) = ?$$

$$\begin{bmatrix} -\lambda & 1+\lambda & 1+\lambda & 1+\lambda & 1+\lambda & \cdots \\ 1 & -\lambda-1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -\lambda-1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & -\lambda-1 & 0 & \cdots \\ 1 & 0 & 0 & 0 & -\lambda-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} (n-1)-\lambda & & & & & \\ 1 & -\lambda-1 & & & & \\ 1 & & -\lambda-1 & & & \\ 1 & & & -\lambda-1 & & \\ 1 & & & & -\lambda-1 & \\ \vdots & & & & & \ddots \end{bmatrix} \qquad \det = \left((n-1)-\lambda\right)(-\lambda-1)^{n-1}$$

$\lambda_1 = n-1$, l. vekt. $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$\lambda_2 = \lambda_3 = \lambda_4 = \ldots = \lambda_n = -1 \leftarrow$ $(n-1)$-kratna l. vred.

Vsak vektor $\begin{bmatrix} \\ -1 \\ \\ 1 \\ \\ \end{bmatrix}$ je l. vekt.
(−1 in 1 sta na poljubnem mestu, drugje so ničle)

Izberemo lahko $(n-1)$ neodv. l. vekt.:

$$\begin{bmatrix} 1 \\ -1 \\ \\ \\ \\ \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ \\ \\ \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ \\ \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}.$$

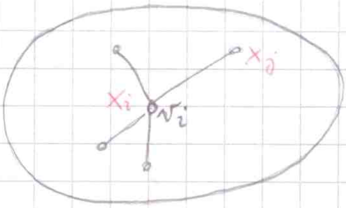
$A(G)x = \lambda x \qquad x = (x_1, \ldots, x_n)^T$

$(A(G))_{ij} = a_{ij} \qquad V(G) = \{v_1, v_2, \ldots, v_n\}$ ... urejena množica

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix} \Rightarrow \sum_{j=1}^{n} a_{ij} x_j = \lambda x_i \quad \forall i$$

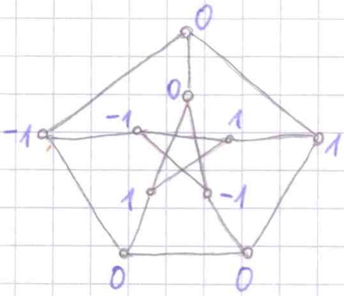
$$a_{ij} = \begin{cases} 1; & v_i \sim v_j \\ 0; & \text{sicer} \end{cases}$$

$$\boxed{\sum_{v_j : v_j \sim v_i} x_j = \lambda x_i}$$



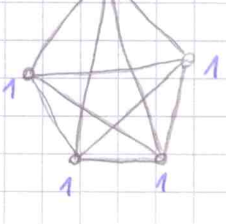
$f : V(G) \rightarrow \mathbb{R}$

Zgled. Petersenov graf.

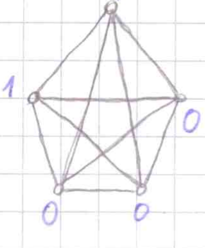


$\lambda = 1$

Zgled. K_5.

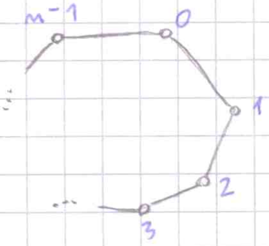


$\lambda = 4$



$\lambda = -1$

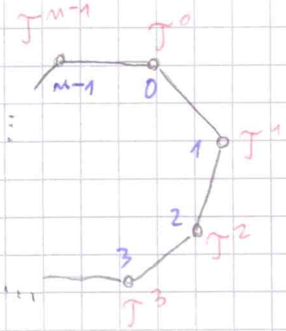
Zgled. C_n



Naj bo ζ n-ti koren enote:

$$\zeta^n = 1.$$

$$\zeta = e^{\frac{2k\pi i}{n}}, \quad 1 \le k \le n.$$



V točki i:

vrednost na sosedih je

$$\zeta^{i+1} + \zeta^{i-1} = \underset{\substack{\uparrow \\ \text{vrednost v } i}}{\zeta^i}(\zeta + \zeta^{-1}) \quad \forall i.$$

$\Rightarrow$ $\zeta + \zeta^{-1}$ je lastna vrednost.

$$\sigma(C_n) = \{\zeta + \zeta^{-1} \mid \zeta^n = 1\}$$

$$\zeta + \zeta^{-1} = e^{\frac{2k\pi i}{n}} + \left(e^{\frac{2k\pi i}{n}}\right)^{-1} =$$

$$= \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) + \left(\cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}\right) =$$

$$= 2\cos\frac{2k\pi}{n}$$

$$\sigma(C_n) = \left\{2\cos\frac{2k\pi}{n} \;\middle|\; 1 \le k \le n\right\}$$

$k = n$: $\lambda = 2$

$G \mapsto A(G) \mapsto S(G)$ ... lastne vred. in lastni vekt.

$A(G)$ je odvisna od vrstnega reda vozlišč.

$A'(G) = P^{-1}A(G)P \leftarrow$ permutacijska matrika

$\Rightarrow$ $\det(A' - \lambda I) = \det(A - \lambda I)$

G k-regularen graf $\Rightarrow$ k je last. vrednost in $\vec{1}$ je pripadajoči last. vektor:

$$k \text{ enic} \longrightarrow \begin{bmatrix} \\ \\ 01 \ldots 10 \ldots 1 \\ \\ \\ \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \\ 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \\ 1 \end{bmatrix} \qquad A\vec{1} = k\vec{1}$$

Izrek. Naj bo G k-regularen graf in naj bodo $k, \lambda_2, \dots, \lambda_n$ njegove lastne vrednosti. Tedaj ima $\overline{G}$ (komplement grafa G) iste lastne vektorje, njegove lastne vrednosti pa so

$$n-k-1, \quad -1-\lambda_2, \quad -1-\lambda_3, \quad \dots, \quad -1-\lambda_n .$$

Dokaz. Najprej opazimo:

$$A(\overline{G}) = J_n - I_n - A(G).$$

$$A(G)\vec{1} = k\vec{1}. \quad A(\overline{G})\vec{1} = (J_n - I_n - A(G))\vec{1} = n\vec{1} - \vec{1} - k\vec{1} = $$
$$= (n-k-1)\vec{1}$$

Naj bo $\frac{\vec{1}}{\|\vec{1}\|}, u_2, u_3, \dots, u_n$ ON baza iz lastnih vektorjev.

$$A(G)u_i = \lambda u_i \qquad 2 \leq i \leq n$$

$$J_n u_i = \begin{bmatrix} \cdots & \vec{1} & \cdots \\ \cdots & \vec{1} & \cdots \\ & \vdots & \\ \cdots & \vec{1} & \cdots \end{bmatrix} u_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A(\overline{G})u_i = (J_n - I_n - A(G))u_i = \vec{0} - u_i - \lambda_i u_i = (-1-\lambda_i)u_i \quad \blacksquare$$

LASTNE VREDNOSTI IN PREMER

$\operatorname{diam}(G) = \max\{d(u,v) \mid u,v \in V(G)\}$

$(A(G)^k)_{ij} = \#$ sprehodov dolžine k med i-tim in j-tim vozliščem v G

A matrika. Minimalni polinom $m(\lambda)$: polinom najnižje stopnje z vodilnim koeficientom 1, tako da je $m(A) = 0$.

Izrek. A simetrična matrika. Tedaj je

$$m(\lambda) = \prod_{i=1}^{t} (\lambda - \lambda_i),$$

kjer so $\lambda_1, \dots, \lambda_t$ različne lastne vrednosti matrike A.

Izrek. Naj bo G poljuben graf. Tedaj je njegov premer manjši od števila različnih lastnih vrednosti grafa G.

K_n : $\mathrm{diam}(K_n) = 1 < 2$ $\quad (\delta(K_n) = \{n-1, -1, \dots, -1\})$

C_n : $\mathrm{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$ $\quad (\delta(C_n) = \{2\cos\frac{2k\pi}{n} ; 1 \leq k \leq n\})$

Dokaz. $k \leq \mathrm{diam}(G)$.

Trdimo: $A(G)^0, A(G)^1, \dots, A(G)^k$ so linearno neodvisne

$A(G)^0 = I$ ✓

$r < k$: $r \to r+1$: $A(G)^0, \dots, A(G)^r$ lin. neodvisne

Naj bosta $u_i, u_j \in V(G)$, tako da je $d(u_i, u_j) = r+1$.

$(r+1 \leq k \leq \mathrm{diam}(G))$

Tedaj je $(A(G)^{r+1})_{ij} \neq 0$ in $(A(G)^l)_{ij} = 0$ za $l = 0, 1, \dots, r$.

Zato $A(G)^{r+1}$ ni linearna kombinacija $A(G)^0, \dots, A(G)^r$.

Naj bo $m(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_t)$ minimalni polinom za $A(G)$. Torej je $\underbrace{(A(G) - \lambda_1 I)(A(G) - \lambda_2 I) \dots (A(G) - \lambda_t I)}_{\text{matrični polinom stopnje } t} = 0$.

$\Rightarrow A(G)^0, A(G)^1, \dots, A(G)^t$ lin. odvisne $\Rightarrow \mathrm{diam}(G) < t$.

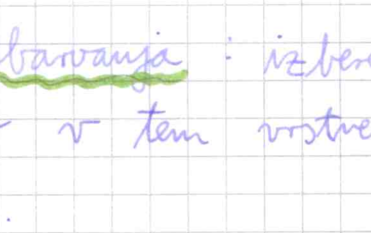
LASTNE VREDNOSTI IN KROMATIČNO ŠTEVILO

$\chi(G)$ = najmanjše št. barv v dobrem barvanju vozlišč grafa G

Požrešni algoritem barvanja : izberemo vrstni red vozlišč ter graf barvamo v tem vrstnem redu z najmanjšimi možnimi barvami.

Posledica algoritma : ocena $\chi(G) \leq \Delta(G) + 1$.

$\uparrow$ maksimalna stopnja točke v grafu



$\lambda_{max}(G)$ ... največja lastna vrednost G

$\lambda_{min}(G)$ ... najmanjša lastna vrednost G

Cilj: $\boxed{\chi(G) \leq \lambda_{max}(G) + 1}$

(Spotoma bomo tudi videli, da je $\lambda_{max} \leq \Delta(G)$.)

Lema 1. Če je A simetrična matrika, tedaj sta vrednosti

$$\max \{ x^T A x \mid \|x\| = 1 \}$$
$$\min \{ x^T A x \mid \|x\| = 1 \}$$

doseženi na lastnih vektorjih, vrednosti pa sta pripadajoči lastni vrednosti.

Lema 2. Naj bo H induciran podgraf grafa G.
Tedaj velja: $\lambda_{min}(G) \leq \lambda_{min}(H) \leq \lambda_{max}(H) \leq \lambda_{max}(G)$.

Dokaz. Izberimo vrstni red vozlišč $V(G)$: $v_1, v_2, \dots, v_n$ tako, da je $V(H) = \{v_1, v_2, \dots, v_j\}$.

$$A(G) = \begin{array}{c@{}c} & \begin{array}{cc} v_1 \dots v_j & v_{j+1} \dots v_n \end{array} \\ \begin{array}{c} v_1 \\ \vdots \\ v_j \\ v_{j+1} \\ \vdots \\ v_n \end{array} & \left[\begin{array}{c|c} A(H) & \\ \hline & \\ & \end{array}\right] \end{array} \begin{bmatrix} z_1 \\ \vdots \\ z_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$z \in \mathbb{R}^j$, $\|z\| = 1 \ni: z^T A(H) z = \lambda_{max}(H)$. Tak z $\exists$ zaradi leme 1.

$$\hat{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} z_1 \\ \vdots \\ z_j \end{matrix}}\right\} z \\ \\ \\ \\ \end{matrix} \in \mathbb{R}^n \qquad \|\hat{z}\| = 1$$

$\lambda_{max}(H) = z^T A(H) z = \hat{z}^T A(G) \hat{z} \leq \lambda_{max}(G)$.

Podobno dokažemo še drugo neenakost. ∎

Zgled. Poišči spekter praznega grafa.

$$\sigma(\overline{K_n}) = ?$$

Že vemo: $\sigma(K_n) = \{n-1, \underbrace{-1, -1, \dots, -1}_{n-1}\}$ in

K_n je $(n-1)$-regularen graf.

↑ polni graf

Po izreku o spektru $\overline{G}$ je

$$\sigma(\overline{K_n}) = \{n-(n-1)-1, -1-(-1), -1-(-1), \dots, -1-(-1)\}$$

$$= \{0, 0, 0, \dots, 0\}$$

Lema 3. Naj bo G graf z n vozlišči in m povezavami. Tedaj velja:

$$\underset{\nwarrow \text{minimalna stopnja}}{\delta(G)} \overset{(a)}{\leq} \frac{2m}{n} \overset{(b)}{\leq} \lambda_{max}(G) \overset{(c)}{\leq} \underset{\nwarrow \text{maksimalna stopnja}}{\Delta(G)}$$

Dokaz. (a) $n \cdot \delta(G) = \sum_{u \in V(G)} \delta(G) \leq \sum_{u \in V(G)} deg(u) = 2m$.

(b)

$x = \frac{\vec{1}_n}{\|\vec{1}_n\|}$, $x \in \mathbb{R}^n$, $\|x\| = 1$.

$x = \frac{1}{\sqrt{n}} \vec{1}_n$

$$\lambda_{max} \overset{\text{Lema 1.}}{\geq} x^T A(G) x = \left(\frac{\vec{1}_n}{\sqrt{n}}\right)^T A(G) \frac{\vec{1}_n}{\sqrt{n}} = \frac{1}{n}\left(\vec{1}_n^{\,T} A(G) \vec{1}_n\right) =$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{i=1}^{n} a_{ij} = \frac{1}{n} \cdot 2m$$

(c) $\lambda_{max}(G) \leq \Delta(G)$

$A(G)x = \lambda x$ $\qquad$ $V(G) = \{u_1, u_2, \dots, u_n\}$

x_i ... največja komponenta lastnega vektorja x

$$\lambda x_i = (\lambda x)_i = (A(G)x)_i = \sum_{j=1}^{n} a_{ij} x_j = \sum_{u_j : u_j \sim u_i} x_j \leq$$

$$\leq \sum_{u_j : u_j \sim u_i} x_i \leq \Delta(G) x_i \Rightarrow \lambda x_i \leq \Delta(G) x_i \Rightarrow$$

$$\Rightarrow \lambda \leq \Delta(G) \ \forall \lambda \Rightarrow \lambda_{max}(G) \leq \Delta(G).$$

■

Def. Graf G je χ-kritičen, če je

$$\chi(G - v) < \chi(G) \qquad (\text{oz.} = \chi(G) - 1).$$

za vsak $v \in V(G)$.

Primeri: K_n: $\chi(K_n) = n$

$\chi(K_n - v) = \chi(K_{n-1}) = n - 1$

C_{2n+1}: $\chi(C_{2n+1}) = 3$

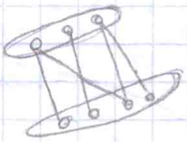
$\chi(C_{2n+1} - v) = \chi(P_{2n}) = 2$

Lema 4. Naj bo G graf s $\chi(G)=k$. Tedaj G vsebuje χ-kritičen podgraf H z $\chi(H)=k$.

Dokaz. Če je G χ-kritičen, tedaj je $H=G$ iskani graf. Sicer (G ni χ-kritičen) G vsebuje $v \in V(G) \ni$:

$$\chi(G-v) = \chi(G).$$

Če je $H=G-v$ χ-kritičen, postavimo $H=G-v$, sicer postopek ponavljamo. ■



Primer. G dvodelni graf $\iff \chi(G)=2$.

$\chi(G)=3$: G ni dvodelen $\Rightarrow$ vsebuje lihi cikel in to je iskani χ-kritičen podgraf.

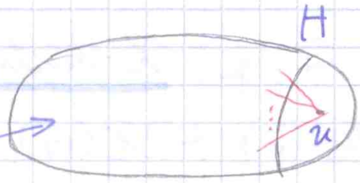
Izrek. Za vsak graf G velja:

$$\chi(G) \le \lambda_{max}(G)+1.$$

Dokaz. Naj bo H χ-kritičen podgraf grafa G z $\chi(H) = \chi(G) = k$ ($\exists$ po lemi 4).

$$\delta(H) \ge k-1 \iff k \le \delta(H)+1 \quad (*)$$

Recimo nasprotno, $\delta(H) \le k-2$ in naj bo u vozlišče iz H stopnje $\le k-2$. Ker je H χ-kritičen, je $\underset{\overset{\shortparallel}{k-1}}{\chi(H-u)} < \chi(H)$.



Lahko pobarvamo s $k-1$ barvami: $\{1,2,\dots,k-1\}$.

Ker je stopnja u-ja $< k-1$, v tej množici obstaja barva, ki se ne pojavi na sosedih od u. Pobarvajmo u s tako manjkajočo barvo in dobimo barvanje H s $k-1$ barvami. $\rightarrow\leftarrow$

$$\chi(G) = k = \chi(H) \overset{(*)}{\leq} \delta(H)+1 \underset{\underset{\text{lema 3}}{\uparrow}}{\leq} \lambda_{max}(H)+1 \underset{\underset{\text{lema 2}}{\uparrow}}{\leq} \lambda_{max}(G)+1.$$

prejšnja ocena

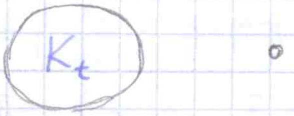
$$\chi(G) \leq \Delta(G)+1 \xrightarrow[\text{izboljšali}]{\text{smo jo}} \chi(G) \leq \lambda_{max}(G)+1$$

Ali morda (v duhu leme 3) velja tudi:

$$\chi(G) \leq \frac{2m}{n} + 1 \ ?$$

Ne! Naj bo G_t disjunktna unija K_t in K_1, $t \geq 2$.

G_t :



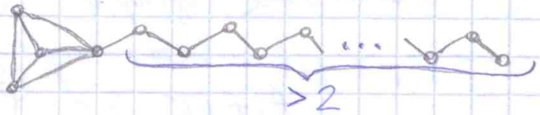
$\chi(G_t) = t$. $\quad \frac{2m}{n} + 1 = \frac{2 \cdot \binom{t}{2}}{t+1} + 1 = \frac{t(t-1)}{t+1} + 1$

$\frac{t(t-1)}{t+1} + 1 < t \iff t(t-1)+t+1 < t(t+1) \iff$

$t^2+1 < t^2+t \iff 1<t$ ✓

Če vzamemo netrivialni primer.

Naloga. Poišči povezan graf, za katerega $\chi(G) \leq \frac{2m}{n}+1$ ne velja.



6. SIMETRIJE GRAFOV

6.1. AVTOMORFIZMI

Def. Avtomorfizem grafa G je izomorfizem $G \to G$.

Tako kot izomorfizmi seveda tudi avtomorfizmi ohranjajo:

- stopnje vozlišč,
- razdalje, ...

$Aut(G) = \{\varphi : G \to G \mid \varphi \text{ avtomorfizem}\}$

- $id \in Aut(G)$
- $\varphi, \psi \in Aut(G) \Rightarrow \varphi\psi \in Aut(G)$