2. KARTEZIČNI IN KREPKI PRODUKT GRAFOV

d' metrika na M', d'' metrika na M''

$M = M' \times M''$ $\quad x \in M: \; x = (x', x'')$

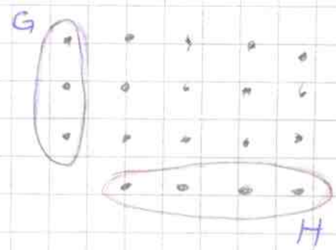
$$\Rightarrow \quad d_M(x,y) = \begin{cases} \left(d'(x',y')^p + d''(x'',y'')^p\right)^{\frac{1}{p}} & ; \; 1 \leq p < \infty \\ \max\{d'(x',y'), d''(x'',y'')\} & ; \; p = \infty \end{cases}$$

$p = 1, p = \infty$

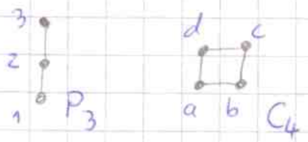
Def. *Kartezični produkt grafov* G in H, $G \square H$, je graf z množico vozlišč $V(G) \times V(H)$ in v katerem je (g,h) sosednje z (g',h') natanko tedaj, ko je bodisi $g = g'$ in $hh' \in E(H)$, bodisi $gg' \in E(G)$ in $h = h'$.

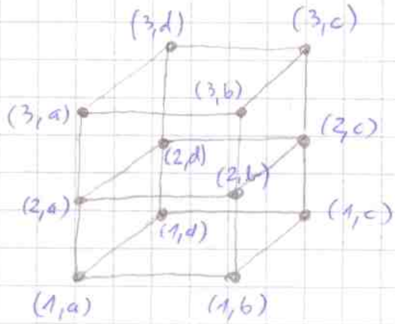
G, H ... faktorja produkta

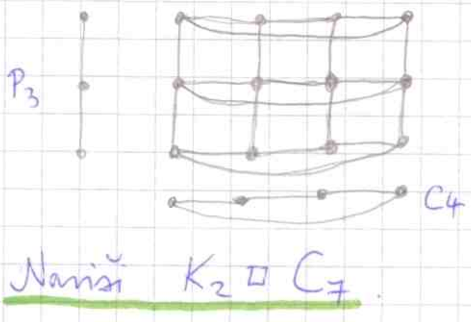
Po navadi rišemo takole:



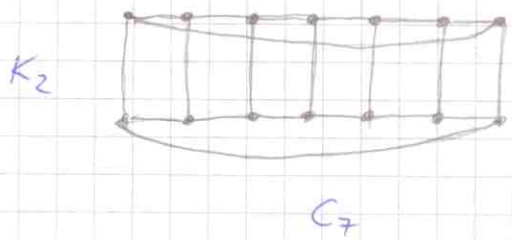
Nariši $P_3 \square C_4$.

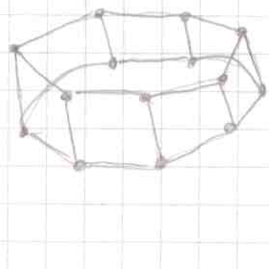




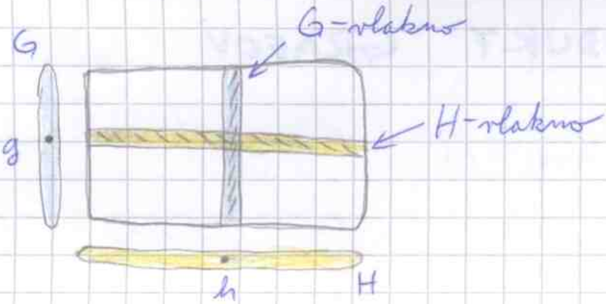


Nariši $K_2 \square C_7$.





$G \square K_2$ ... prizma nad G



$g \in G$: $\{(g,h) \mid h \in V(H)\}$ je H-vlakno. H-vlakno je izomorfno H.

$\square$: "$K_2 \square K_2 =$  "

<u>Trditev</u>. Operacija $\square$ je asociativna in komutativna. Graf K_1 je enota za kartezično množenje grafov.

<u>Ideja dokaza</u>: $(G \square H) \square K$ graf izomorfen grafu $G \square (H \square K)$

Preverimo, da je $((g,h),k) \longmapsto (g,(h,k))$ izomorfizem grafov. Za komutativnost moramo preveriti, da je $(g,h) \longmapsto (h,g)$ izomorfizem. K_1 je enota: $K_1 \square G$ je izomorfen G.

Zaradi asociativnosti je $G_1 \square G_2 \square \dots \square G_k$ dobro definirana operacija. Pišemo tudi $\mathop{\square}\limits_{i=1}^{k} G_i$.

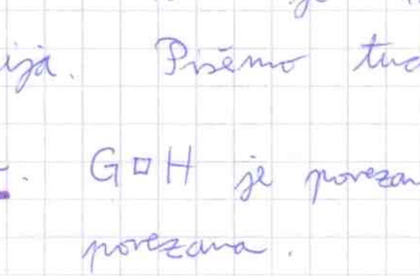
<u>Trditev</u>. $G \square H$ je povezan natanko tedaj, ko sta G in H povezana.

<u>Dokaz</u>. G, H povezana.

$(g,h), (g',h') \in V(G \square H)$

V G obstaja pot P med g in g'.

V H obstaja pot Q med h in h'.



$P: g \to g_1 \to \dots \to g_r \to g'$

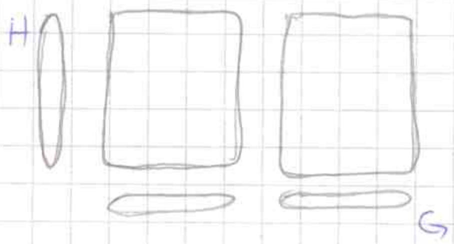
$Q: h \to h_1 \to \dots \to h_s \to h'$

$(g,h) \to (g_1,h) \to \dots \to (g_r,h) \to$

$\to (g',h) \to (g',h_1) \to \dots \to$

$\to (g', h_2) \to (g', h')$. To je pot med (g,h) in (g',h') v $G \square H$.

Obratno: Recimo, da G ni povezan. Potem pa tudi produkt ni povezan:

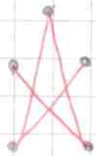


<u>Trditev.</u> Naj bo $X = G \square H$. Tedaj je $d_X((g,h),(g',h')) = d_G(g,g') + d_H(h,h')$.

Zaradi asociativnosti takoj velja: $G = \square_{i=1}^{k} G_i \Rightarrow d_G = \sum_{i=1}^{k} d_{G_i}$.

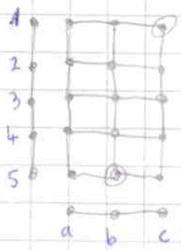
$\bar{G}$ ... **komplement** grafa G

$V(\bar{G}) = V(G)$, $uv \in E(\bar{G}) \iff uv \notin E(G)$.

G:  $\bar{G} = P_5$ 

$\overline{C_5} = C_5$, C_5 je sebi komplementaren.

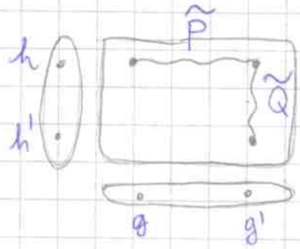
<u>Dokaz trditve.</u> $d_X \le d_G + d_H$



Vzemimo poljubni vozlišči (g,h) in (g',h') iz X.

P ... najkrajša pot med g in g' v G, $d_G(g,g') = |P|$

Q ... —||— h in h' v H, $d_H(h,h') = |Q|$

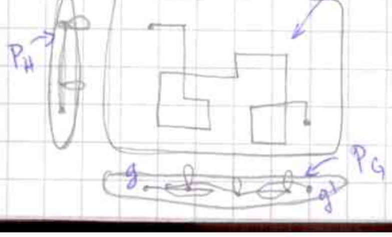


$$d_X((g,h),(g',h')) \le |\tilde{P}| + |\tilde{Q}| = |P| + |Q| = d_G(g,g') + d_H(h,h')$$

$\tilde{P}$ ... pot med (g,h) in (g',h) v G-vlaknu

$d_G + d_H \le d_X$

P ... najkrajša pot med (g,h) in (g',h'), $|P| = d_X((g,h),(g',h'))$



P_H ... projekcija poti P na H; P_H ni nujno pot, v splošnem je le sprehod

povezave so samo horizontalne ali vertikalne

$$|P| = |P_G| + |P_H| \ge d_G(g,g') + d_H(h,h')$$

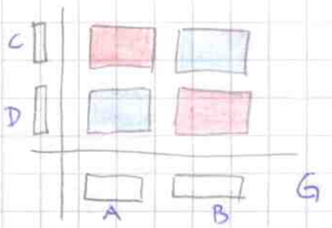
Naloga. Dokaži, da je kartezični produkt dveh dvodelnih grafov spet dvodelni graf.

G, H dvodelna grafa $\Rightarrow$ $G \square H$ dvodelen graf

G je **dvodelen**, če $\exists A, B \subseteq V(G) \ni: A \cap B = \emptyset$ in $A \cup B = V(G)$, pri čemer velja: $uv \in E(G) \Rightarrow u \in A$ in $v \in B$ ali obratno

$$\left.\begin{array}{l} V(G) = A + B \\ V(H) = C + D \end{array}\right\} V(G \square H) = A \times C + A \times D + B \times C + B \times D$$

(g,h) in $(g',h') \in A \times C \cup B \times D$



Nista sosednji:

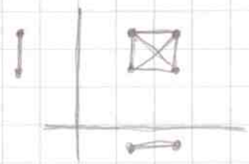
1. $g = g'$ in $hh' \in E(H)$
   1.1. $g \in A \Rightarrow h$ in $h' \in C \Rightarrow \rightarrow\leftarrow$, saj ni povezave med h in h'
   1.2. $g \in B \Rightarrow h$ in $h' \in D \Rightarrow \rightarrow\leftarrow$, saj ni povezave
2. analogno: $h = h'$ in $gg' \in E(G)$
   2.1. $h \in C \Rightarrow g$ in $g' \in A \Rightarrow \rightarrow\leftarrow$
   2.2. $h \in D \Rightarrow g$ in $g' \in B \Rightarrow \rightarrow\leftarrow$

Def. **Krepki produkt**, $G \boxtimes H$, grafov G in H je graf z $V(G \boxtimes H) = V(G) \times V(H)$ in v katerem je $(g,h) \sim (g',h')$, če velja eden izmed pogojev:

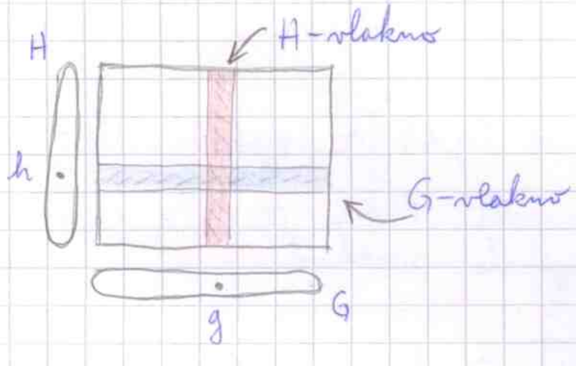
(i) $g = g'$ in $hh' \in E(H)$

(ii) $gg' \in E(G)$ in $h = h'$

(iii) $gg' \in E(G)$ in $hh' \in E(H)$

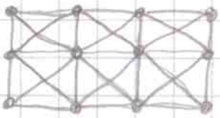


$K_2 \boxtimes K_2 = K_4$



Trditev. Krepki produkt je asociativen in komutativen, K_1 je enota za to množenje. $G \boxtimes H$ je povezan graf natanko tedaj, ko sta G in H povezana grafa.

Zgled. $P_3 \boxtimes P_4$:

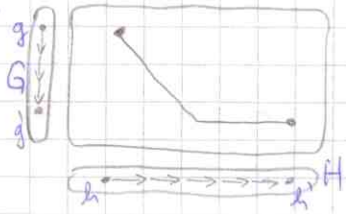


Trditev. Naj bo $X = G \boxtimes H$. Tedaj je $d_X((g,h),(g',h')) = \max\{d_G(g,g'), d_H(h,h')\}$.

Najprej $d_X \le \max\{d_G, d_H\}$:

Dokaz. $g \to g_1 \to \dots \to g_r \to g'$

$h \to h_1 \to \dots \to h_r \to \dots \to h_s \to h'$



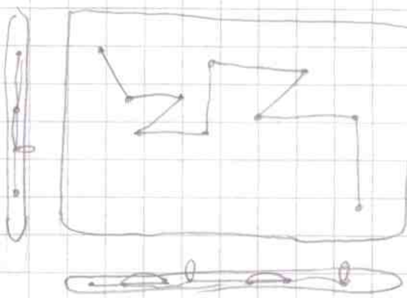
Brez izgube splošnosti: $r \le s$.

$(g,h) \to (g_1,h_1) \to \dots \to (g_r,h_r) \to (g',h_{r+1}) \to \dots \to (g',h_s) \to (g',h')$

je pot med (g,h) in (g',h'), katere dolžina je $d_H(h,h')$.

$\max\{d_G, d_H\} \le d_X$

Še obrat: Recimo, da imamo krajšo pot. (tj. krajšo od $\max\{d_G, d_H\}$). BŠS: $d_G \ge d_H$, torej $d_X < d_G$. Projekcija te poti na G bi bil sprehod med g in g', ki je krajši od najkrajše poti med g in g' $\rightarrow \leftarrow$ □



Izrek. Vsak povezan graf lahko izometrično vložimo v krepki produkt poti.

Dokaz. G, $V(G) = \{v_1, v_2, \dots, v_n\}$

$$e(v_i) = \max_{x \in V(G)} d(v_i, x) + 1 \qquad 1 \le i \le n-1$$

$$\beta : G \to P_{e(v_1)} \boxtimes P_{e(v_2)} \boxtimes \dots \boxtimes P_{e(v_{n-1})} =: H$$

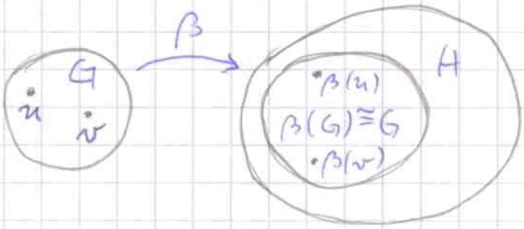
$$\beta(v_i) = (\beta_1(v_i), \beta_2(v_i), \dots, \beta_{n-1}(v_i))$$

kjer je $\beta_j(v_i) = d_G(v_j, v_i)$

Pri tem privzamem, da je $V(P_k) = \{0, 1, \dots, k-1\}$.

Trdim: β je izometrična vložitev

$\beta : G \to H$



$$d_H(\beta(u), \beta(v)) = d_G(u, v) \quad \forall u, v \in V(G).$$

(a) β slika vozlišča G v vozlišča H

$$\beta_j(v_i) \leq e(v_i) \Rightarrow \beta_j(v_i) \in V(P_{e(v_i)})$$

(b) β ohranja razdaljo

$v_k, v_\ell \in V(G)$, $k \leq n-1$

lahko bi predp. tudi ($v_k \neq v_\ell$, sicer ni zanimivo)

$$\beta(v_k) = (\beta_1(v_k), \dots, \beta_k(v_k), \dots, \beta_{n-1}(v_k))$$
$$\beta(v_\ell) = (\beta_1(v_\ell), \dots, \beta_k(v_\ell), \dots, \beta_{n-1}(v_\ell))$$

$$\beta_k(v_k) = d(v_k, v_k) = 0$$
$$\beta_k(v_\ell) = d(v_k, v_\ell)$$

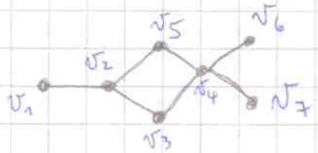
$$\begin{aligned} d_H(\beta(v_k), \beta(v_\ell)) &= \max_i \{ d_{P_{e(v_i)}}(\beta_i(v_k), \beta_i(v_\ell)) \} \\ &\geq d_{P_{e(v_k)}}(\beta_k(v_k), \beta_k(v_\ell)) \\ &= |\beta_k(v_\ell) - \beta_k(v_k)| = | d_G(v_k, v_\ell) - 0 | \\ &= d_G(v_k, v_\ell) \end{aligned}$$

Pokazati moramo še, da razdalja v H ne more biti večja, kot je v G. Naj bo i poljubna druga koordinata.

BŠS: $\quad d_G(v_i, v_k) \geq d_G(v_i, v_\ell)$.

$$\begin{aligned} d_{P_{e(v_i)}}(\beta_i(v_k), \beta_i(v_\ell)) &= | d_G(v_i, v_k) - d_G(v_i, v_\ell) | = \\ &= d_G(v_i, v_k) - d_G(v_i, v_\ell) \underset{\text{trikotniška neenakost}}{\leq} d_G(v_k, v_\ell). \end{aligned}$$

Zgled.

G: 

i	$e(v_i)$
1	$4+1$
2	$3+1$
3	$2+1$
4	$3+1$
5	$2+1$
6	$4+1$
~~7~~	~~$4+1$~~

ne rabimo

$$\beta : G \longrightarrow P_5 \boxtimes P_4 \boxtimes P_3 \boxtimes P_4 \boxtimes P_3 \boxtimes P_5 =: H$$

$\beta(v_1) = (0,1,2,3,2,4)$

$\beta(v_2) = (1,0,1,2,1,3)$

$\beta(v_3) = (2,1,0,1,2,2)$

$\beta(v_4) = (3,2,1,0,1,1)$

$\beta(v_5) = (2,1,2,1,0,2)$

$\beta(v_6) = (4,3,2,1,2,0)$

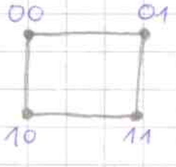
$\beta(v_7) = (4,3,2,1,2,2)$

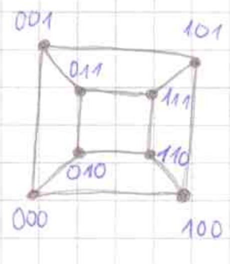
$d_H(\beta(v_2), \beta(v_4)) =$
$= \max\{|1-3|, |0-2|, |1-1|, |2-0|, |1-1|, |3-1|\} = 2$

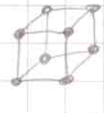
3. METRIČNO DEFINIRANI RAZREDI GRAFOV

Def. n-kocka, Q_n, $(n \geq 1)$ je graf z vozlišči $V(Q_n) = \{0,1\}^n$, v katerem sta vozlišči sosednji, če se razlikujeta v natanko eni koordinati.

Q_1: 

Q_2: 

Q_3: 



Trditev. Q_d je d-regularen graf na 2^d vozliščih in $d \cdot 2^{d-1}$ povezavah.

Dokaz.

$b_1 b_2 \dots b_d$

sosedje: $\underbrace{\overline{b_1} b_2 \dots b_d \quad b_1 \overline{b_2} b_3 \dots b_d \quad b_1 b_2 \dots \overline{b_d}}_{\# = d}$

$|V(Q_d)| = |\{0,1\}^d| = 2^d$.